Parameterized aspects of strong subgraph closure

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A relaxation of F-Free Edge Deletion

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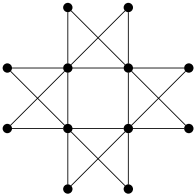

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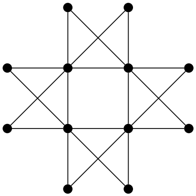

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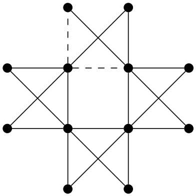

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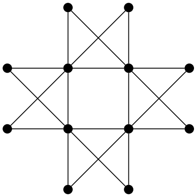

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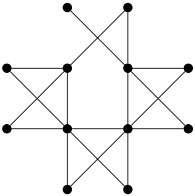

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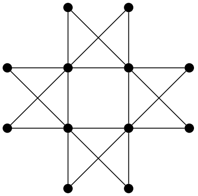

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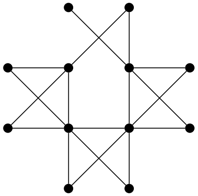

C_4-free Deletion(G)

- Relaxation:

  for every $S \subseteq V_H$:

  $$\boxed{H[S] = F \Rightarrow G[S] \neq F}$$

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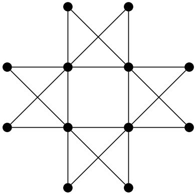

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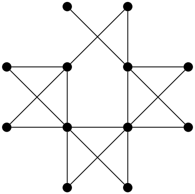

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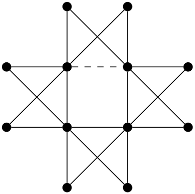

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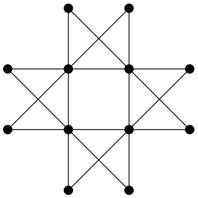

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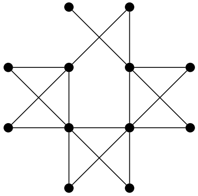

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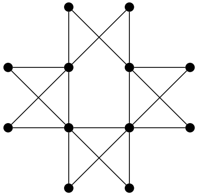

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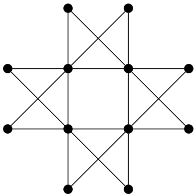

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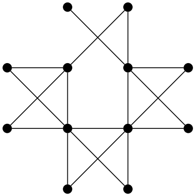

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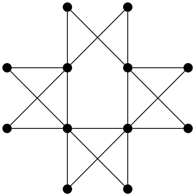

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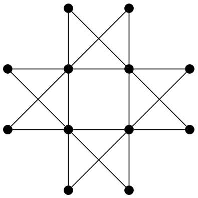

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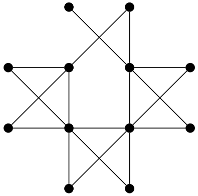

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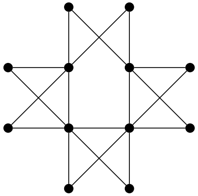

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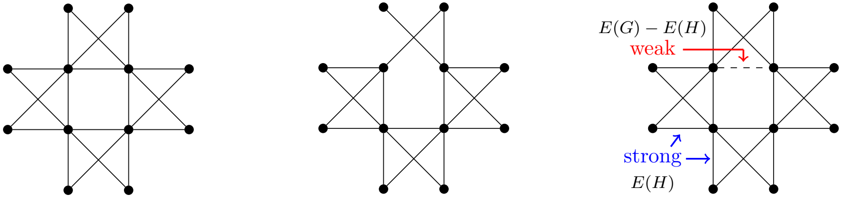

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Label the edges strong and weak
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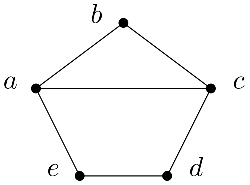

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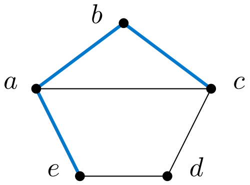

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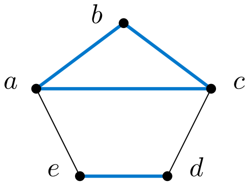

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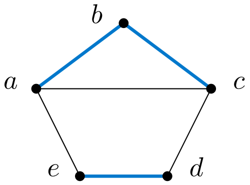

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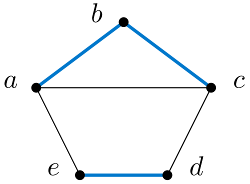

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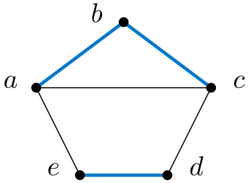

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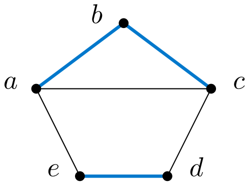

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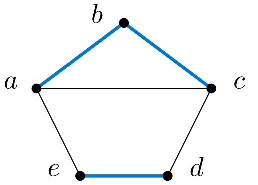

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- STC is polynomial solvable in proper interval graphs, cographs, and graphs of bounded treewidth

Our work

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We study STRONG F-CLOSURE from a parameterized complexity point of view.

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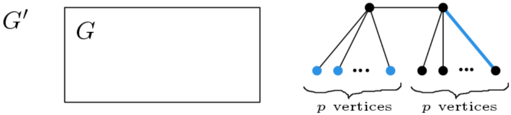

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Case 1. F has a connected component with at least 3 vertices.
Case 2. $F = pK_1 + qK_2$, with $p \geq 0$ and $q \geq 2$.

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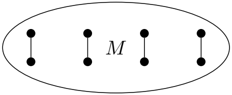

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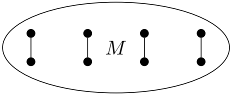

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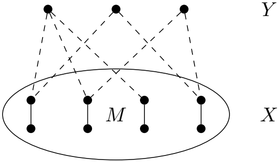
Otherwise,
$X = V_M$ with $|X| \leq 2k - 2$
$Y = V_G \setminus X$ is an independent set



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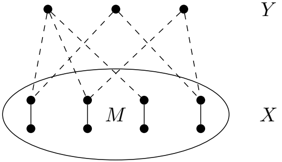
If F has a connected component with at least 3 vertices, Strong F-Closure has a kernel.

Rule 1. If there is a set of $|V_F| + k + 1$ false twins in G, then remove one of them.

Find maximal matching M.
Note that $E_H = M$ satisfies Strong F-closure. If $|M| \geq k$ then this is a solution.

Otherwise,
$X = V_M$ with $|X| \leq 2k - 2$
$Y = V_G \setminus X$ is an independent set

At most $2^{|X|}$ vertices of Y with distinct neighborhoods.
Every false twin class has size at most $|V_F| + k$.



Our work

Parameter: $|E_H| + |V_F|$ (strong edges + size of F)

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Theorem

If $F \neq pK_1$ with $p \geq 1$ and $F \neq pK_1 + K_2$ with $p \geq 0$, then Strong F-Closure is FPT parameterized by $|E_H| + |V_F|$.

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If $|E_F| \leq k$ and F has no isolated vertices, $|V_F| \leq k$.

Corollary

If F has no isolated vertices, Strong F-Closure is FPT parameterized by $|E_H|$, even when F is given as part of the input.

Our work

Parameter	Restriction	Parameterized Complexity
$\|E_H\| + \|V_F\|$	$\|E_F\| \leq 1$	co-$W[1]$-hard
	$\|E_F\| \geq 2$	FPT
	F has a component with ≥ 3 vertices, G is d-degenerate	polynomial kernel
$\|E_H\|$	F has no isolated vertices	FPT
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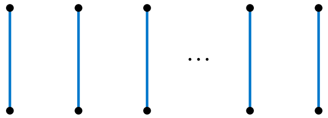
F has a connected component with at least three vertices:

If H is a matching, then H satisfies the F-Closure.



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$\lvert E_H\rvert - \nu(G)$	$F = P_3$, $\Delta(G) \leq 4$	FPT
	$F = K_{1,t}$, $t \geq 3$	FPT

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For every fixed graph F, Strong F-closure can be solved in time $2^{O(\ell)} \cdot n^{O(1)}$, where $\ell = |E_G| - |E_H|$.

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1. List all induced subgraphs of G isomorphic to F.
   - F fixed $\rightarrow$ poly-time.

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Theorem

For every fixed graph F, Strong F-closure can be solved in time $2^{O(\ell)} \cdot n^{O(1)}$, where $\ell = |E_G| - |E_H|$.

1. List all induced subgraphs of G isomorphic to F.
   - F fixed $\rightarrow$ poly-time.
2. For each induced subgraph $F' \simeq F$ we check whether $G[V_{F'}]$ has a weak edge.
   - If it does not, then we must make at least one of the edges of $G[V_{F'}]$ weak.
   - Branch.

Our work

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$\|E_G\| - \|E_H\|$	None	FPT
		poly generalized kernel

Thank you! ☺