Incompressibility of H-free edge modification problems: Towards a dichotomy

Sandeep R. B.
IIT Dharwad, India
Work in progress with **Dániel Marx**
Max Planck Institute for Informatics, Germany

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H-free edge modification problems

Definition

For a graph H, an H-free edge modification problem is to check whether there exist at most k edges in the input graph such that modifying them results in a graph without any induced copy of H.

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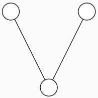
Types:

- H-free Edge Deletion
- H-free Edge Completion
- H-free Edge Editing

An Example: P_3-FREE EDGE EDITING



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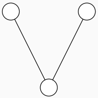

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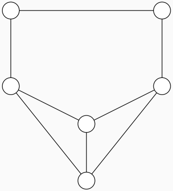
Do there exist at most 3 edges or non-edges in the following graph, editing which results in a P_3-free graph?



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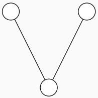

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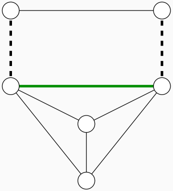

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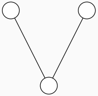

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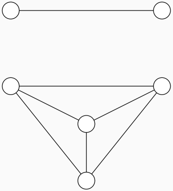

NP-completeness

A dichotomy was obtained by Aravind, Sandeep, and Sivadasan [SIDMA, 2017].

	P	NP-complete
H-free Edge Editing	$\lvert V(H)\rvert < 3$	$\lvert V(H)\rvert \geq 3$
H-free Edge Deletion	$\lvert E(H)\rvert < 2$	$\lvert E(H)\rvert \geq 2$
H-free Edge Completion	$\lvert \overline{E}(H)\rvert < 2$	$\lvert \overline{E}(H)\rvert \geq 2$

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Folklore:

- H-FREE EDGE EDITING is polynomial-time solvable if and only if $\overline{H}$-FREE EDGE EDITING is polynomial-time solvable
- H-FREE EDGE DELETION is polynomial-time solvable if and only if $\overline{H}$-FREE EDGE COMPLETION is polynomial-time solvable.

Parameterized complexity

- Parameter: k
- All H-free edge modification problems are in FPT [Cai, IPL, 1996]
- A simple branching algorithm has a complexity $O^*(2^{O(k)})$
- But no parameterized subexponential-time ($O^*(2^{o(k)})$) algorithm for the hard cases (assuming the ETH) [SIDMA, 2017]

Polynomial kernelization

- First incompressibility results by Kratsch and Wahlström[Discrete Optimization, 2013]
- Completely settled: Paths, Cycles and 3-connected graphs [Cai and Cai, Algorithmica, 2015]
- Diamond -  [Cao, Rai, Sandeep, Ye, ESA 2018]
- Paw -  [Eiben, Lochet, Sourabh, 2019; Cao, Ke, Yuan, 2019]

Our results

- A complete dichotomy for regular graphs H: For a regular graph H, H-free edge modification problems do not admit polynomial kernels if and only if H is neither empty nor complete (incompressibility assumes NP $\not\subseteq$ coNP/poly).

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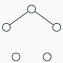
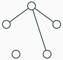
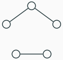
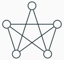
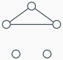
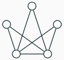
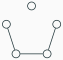
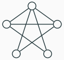
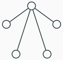
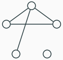
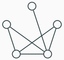
- A complete dichotomy for regular graphs H: For a regular graph H, H-free edge modification problems do not admit polynomial kernels if and only if H is neither empty nor complete (incompressibility assumes NP $\not\subseteq$ coNP/poly).
- A conditional dichotomy for H-free Edge Editing: For a graph H with at least five vertices, H-free Edge Editing admits no polynomial kernel if and only if H is neither complete nor empty, provided the problem does not admit polynomial kernel when H is one in the set of ~~20~~ 14 graphs (shown in the next slide).

The gang of twenty

#	H	$\overline{H}$
1		
2		
3		
4		
5		
6		
7		

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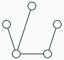
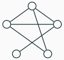
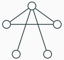
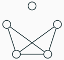
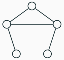
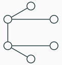
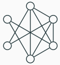
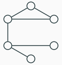
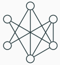
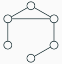
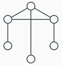
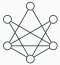
8		
9		
10		same
11		
12		
13		
14		

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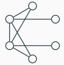
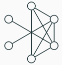
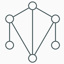
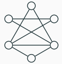
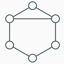
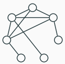
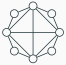
15		
16		
17		
18		
19		same
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Polynomial Parameter Transformation (PPT)

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- If there is a PPT from a problem P to Q and if P is incompressible (no polynomial kernel exists) then so is Q.
- For every graph H we give a sequence of PPTs from H'-free Edge Editing to H-free Edge Editing, where H' is among the 20 graphs or the corresponding problem is known to be incompressible.

Regular graphs

For a connected graph G, we say that G has a non-separating subgraph G', if there exist a vertex set $U \subseteq V(G)$ such that U induces G' in G and $G - U$ is connected.

Lemma

Let G be a connected regular graph which is not complete. Then either of the following two statements holds true:

(i) *G has a non-separating $2K_2$, C_4 or C_5.*

(ii) *$\overline{G}$ is connected and has a non-separating C_4.*

Regular graphs...

- For a connected r-regular graph H, which is not complete, we give a PPT from H'-free Edge Editing to H-free Edge Editing where H' is either a $2K_2$, C_4, or C_5.

Regular graphs...

- For a connected r-regular graph H, which is not complete, we give a PPT from H'-free Edge Editing to H-free Edge Editing where H' is either a $2K_2$, C_4, or C_5.
- By a result in [SIDMA 2017], if the incompressibility for connected regular graphs implies that of all regular graphs which are neither complete nor empty.

A Construction

Taken from [SIDMA, 2017]
Input: Graphs: G' and H; $V' \subseteq V(H)$; integer k.
Output: Graph G.
Let H' be $H[V']$.

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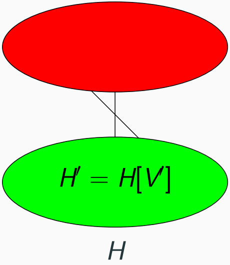

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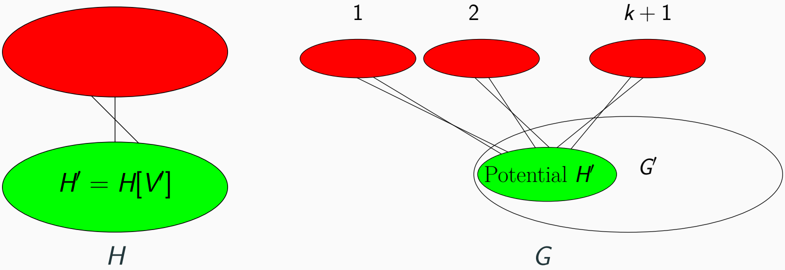

Identifying the gang

Theorem

*There exists a set $\mathcal{H}$ of twenty graphs such that if H-*FREE EDGE EDITING *is incompressible for every $H \in \mathcal{H}$, then H-*FREE EDGE EDITING *is incompressible for every H having at least five vertices but is neither complete nor empty.*

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$$\begin{aligned}
\mathcal{X} = \{ & C_\ell, C_\ell \text{ for all } \ell \geq 4, \\
& P_\ell, \overline{P_\ell} \text{ for all } \ell \geq 5, \\
& H \text{ such that either } H \text{ or } \overline{H} \text{ is 3-connected but not complete} \\
& H \text{ such that } H \text{ is regular but is neither complete nor empty}\} \\
\mathcal{Y} = \{ & K_t, \overline{K_t} \text{ for all } t \geq 1, \\
& P_3, \overline{P_3}, P_4, \\
& \text{diamond}, \overline{\text{diamond}}, \text{paw}, \overline{\text{paw}}, \text{claw}, \overline{\text{claw}}\}
\end{aligned}$$

Identifying the gang...

- *H is reducible to H'* := there is a PPT from H'-free Edge Editing to H-free Edge Editing.
- A set of graphs $\mathcal{H}$ is called base for a set $\mathcal{G}$ of graphs, if every graph $H \in \mathcal{G}$ can be reduced to a graph $H' \in \mathcal{H} \cup \mathcal{X}$.
- The objective is to find a base for all graphs with at least five vertices which are neither complete nor empty.

Churning H

Proposition (SIDMA 2017)

Let H' be obtained from H by deleting all lowest (V_ℓ) or highest (V_h) degree vertices. Then H is reducible to H'.

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Let H' be obtained from H by deleting all lowest (V_ℓ) or highest (V_h) degree vertices. Then H is reducible to H'.

Churn(H): // H is a graph with at least five vertices and is neither complete nor empty.

Step 1: If $H \in \mathcal{X}$, then return H.

Step 2: If $H - V_\ell \notin \mathcal{Y}$, then $H = H - V_\ell$ and goto Step 1.

Step 3: If $H - V_h \notin \mathcal{Y}$, then $H = H - V_h$ and goto Step 1.

Step 4: Return H.

Case analysis

$H - V_\ell$ \ $H - V_h$	Complete	Empty
Complete	$\{C_4, H_6\}$	$\{H_4, H_6, H_{10}, H_{11}, H_{14}, H_{18}, H_{19}, H_{20}\}$
P_3	$\emptyset$	$\{C_4, H_1\}$
P_4	$\emptyset$	$\{H_5\}$
claw	$\emptyset$	$\{H_2\}$
paw	$\emptyset$	$\{H_7, H_{17}\}$
diamond	$\{C_4, H_3\}$	$\{H_3, H_8, H_9, H_{15}, H_{16}\}$

Case analysis...

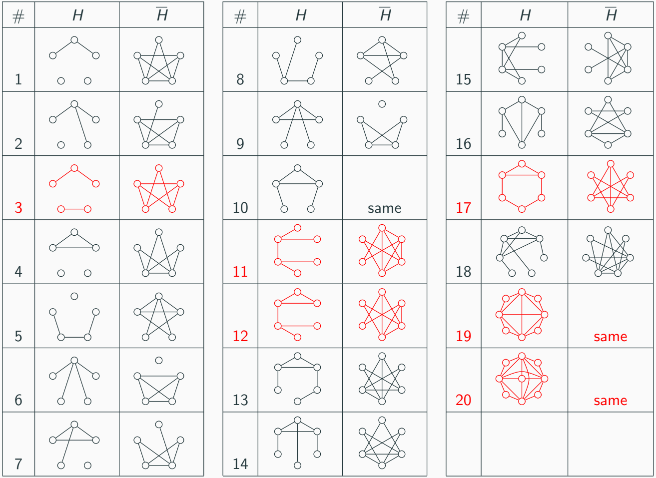
$H - V_\ell$ \ $H - V_h$	P_3	P_4	claw	paw	diamond
P_3	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
P_4	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
claw	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
paw	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
diamond	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

$H - V_\ell$ \ $H - V_h$	$\overline{P_3}$	P_4	$\overline{\text{claw}}$	$\overline{\text{paw}}$	$\overline{\text{diamond}}$
P_3	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
P_4	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
claw	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
paw	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\{H_{12}, H_{13}\}$
diamond	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\{H_3\}$

$H - V_\ell$ \ $H - V_h$	P_3	P_4	claw	paw	diamond
$\overline{P_3}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
P_4	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\overline{\text{claw}}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\overline{\text{paw}}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\overline{\text{diamond}}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

Taming the gang

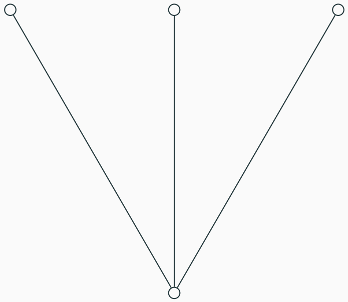
Cai and Cai [Algorithmica, 2015] has a general PPT from a satisfiability problem, which can be used to prove the incompressibility of 6 among the gang.



Open problem: Incompressible?

Eliminate the gang!

Open problem: Polynomial Kernels?



Future problems: Deletion and Completion

Obtain similar sets of base graphs for H-free Edge Deletion and H-free Edge Completion.

Future problems: Complexity of $\{H_1, H_2\}$-free edge modification problems

Obtain complexity dichotomy (P or NP-hard) for $\{H_1, H_2\}$-free edge modification problems.

Thank You!