Polynomial Kernels for Paw-free Edge Modification Problems

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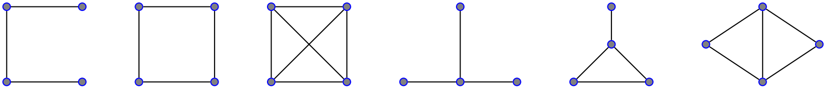
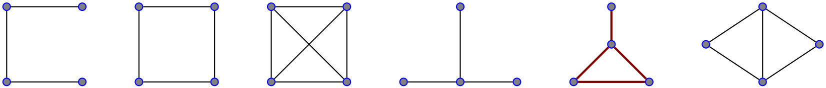
Workshop on Graph Modification, Bergen

H-free graphs



Graphs on four vertices

H-free graphs



H	completion	deletion	editing
K_4	*trivial*	$O(k^3)$	$O(k^3)$ [Dekel Tsur 2019]
P_4	$O(k^3)$	$O(k^3)$	$O(k^3)$ [Sylvain Guillemot et al. 2013]
diamond	*trivial*	$O(k^3)$	$O(k^8)$ [Yixin Cao et al. 2018]
paw	$O(k)$	$O(k^4)$ [this graph]	$O(k^6)$ [next talk]
claw	*unknown*	*unknown*	*unknown*
C_4	*no*	*no*	*no* [Sylvain Guillemot et al. 2013]

Paw-free completion problem

Paw-free completion

Input: A graph G, an integer k.

Task: An edge set E_+ of size at most k such that $G + E_+$ is a paw-free graph.

Our result: A $38k$-vertex kernel

Paw-free edge deletion problem

Paw-free deletion

Input: A graph G, an integer k.
Task: An edge set E_- of size at most k such that $G - E_-$ is a paw-free graph.

Our result: An $O(k^4)$-vertex kernel

Paw-free Completion

Paw-free graphs

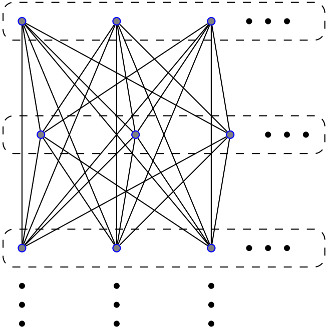
Proposition [Stephan Olariu 1988]

A graph G is paw-free iff every component of G is triangle-free or complete multipartite.

Paw-free graphs

Proposition [Stephan Olariu 1988]

A graph G is paw iff every component of G is triangle-free or complete multipartite.



complete multipartite

Main ideas

- Produce a *modulator* M, where $|M| \leq 4k$.
  M is a modulator of G if every paw of G intersects M by at least two vertices

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- $G - M$ is paw-free, every component of $G - M$ is $\triangle$-free or complete multipartite.

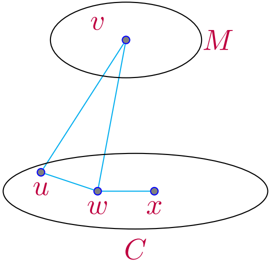
- The number of vertices in $\triangle$-free components of $G - M$ is $O(k)$.
  A triangle-free component of $G - M$ is of type I if it forms a triangle with a vertex in M, or type II otherwise

Main ideas

- Produce a *modulator* M, where $|M| \leq 4k$.
- $G - M$ is paw-free, every component of $G - M$ is $\triangle$-free or complete multipartite.
- The number of vertices in $\triangle$-free components of $G - M$ is $O(k)$.
- The number of vertices in complete multipartite components of $G - M$ is $O(k)$.

Triangle-free components

Let M be a modulator of G, and C a $\triangle$-free component of $G - M$.



Proposition

If $v \in M$ forms a triangle with an edge in C, then

(i) v is adjacent to all the vertices of C;

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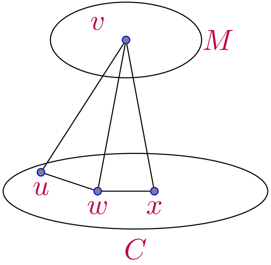

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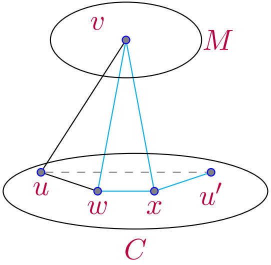

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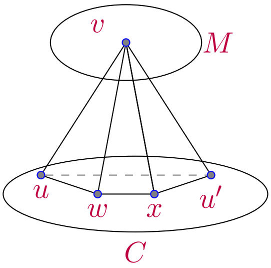

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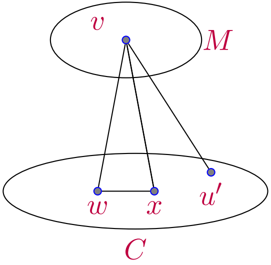

$\longrightarrow$ C is $\{K_3, \overline{P_3}\}$-free

Proposition

If $v \in M$ forms a triangle with an edge in C, then

(i) v is adjacent to all the vertices of C;

(ii) C is complete bipartite.

Triangle-free components

Every type I $\triangle$-free component is complete bipartite.

Multipartite components

Proposition

M is a modulator of G. If a vertex $v \in M$ adjacent to a multipartite component C of $G - M$, then $C \cap \overline{N(v)}$ is either empty or precisely one part of C.

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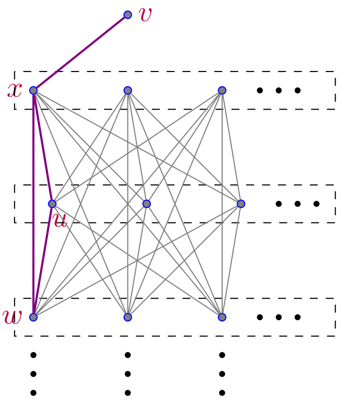

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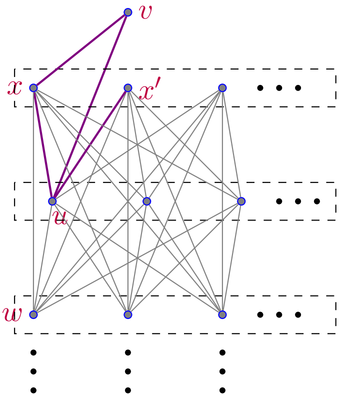

Multipartite components

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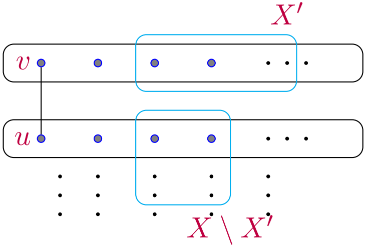
M is a modulator of G. If a vertex $v \in M$ adjacent to a multipartite component C of $G - M$, then $C \cap \overline{N(v)}$ is either empty or precisely one part of C.

$\longrightarrow$ Each part of C is a false twin class of G

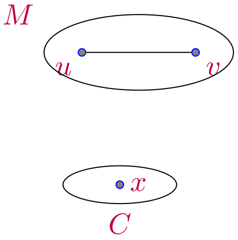
Trivial components

Let G be a connected graph containing a paw and uv an edge in G. We need to add at least $|V(G) \setminus N[\{u,v\}]|$ edges incident to u or v to G to make it paw-free.



$X = V(G) \setminus N[\{u,v\}]$

Trivial components



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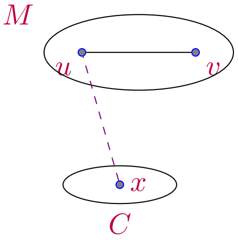

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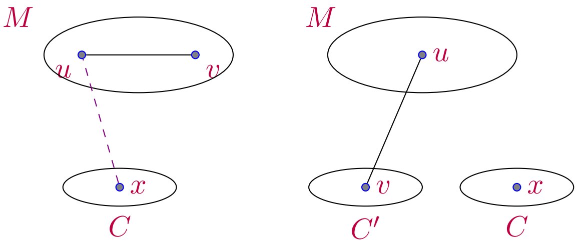

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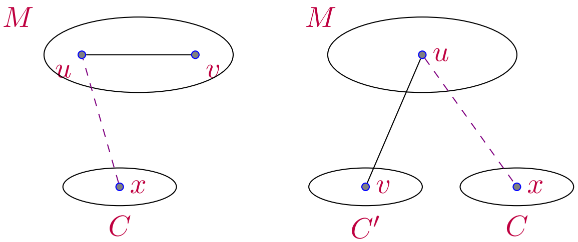

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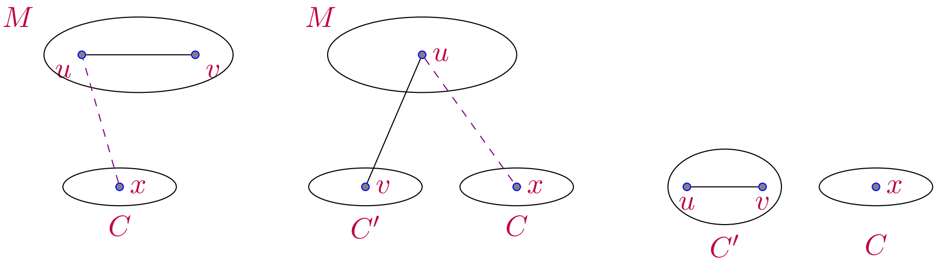

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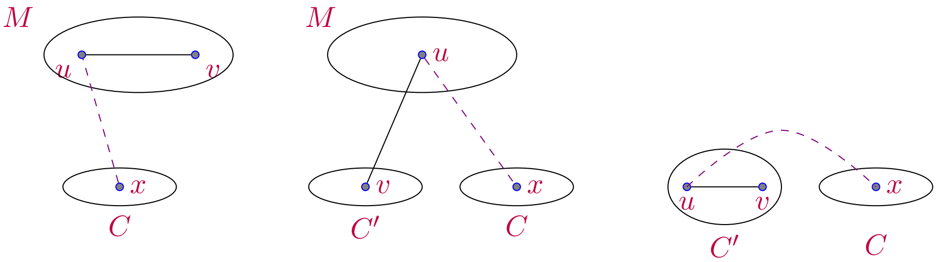

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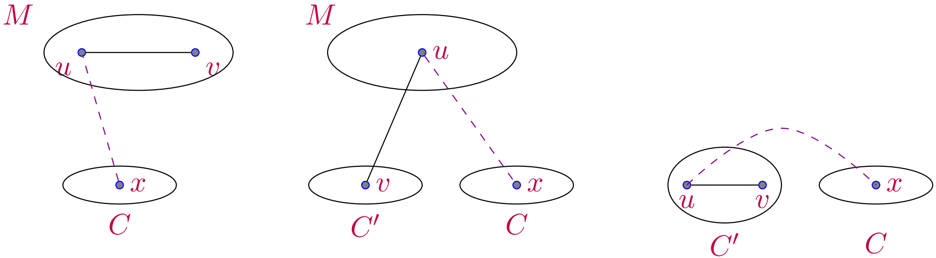
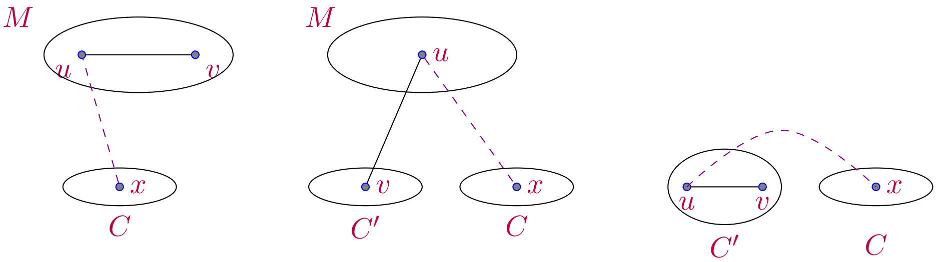


At most $2k$ such trivial components of $G - M$

Trivial components

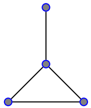


At most $2k$ such trivial components of $G - M$

Every other isolated vertex in $G - M$ dominating all edges in G'

Modulator

Construction of the modulator M



1. each paw F in G, if $|F \cap F'| \leq 1$ for each paw F' in M

Modulator

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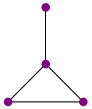

1. each paw F in G, if $|F \cap F'| \leq 1$
for each paw F' in M

add all vertices of F to M

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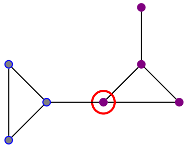

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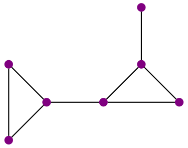

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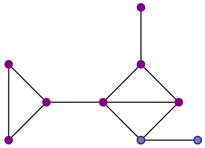

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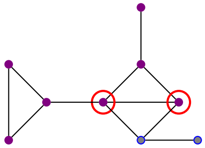

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Modulator

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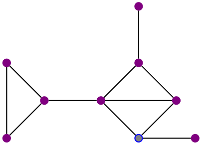
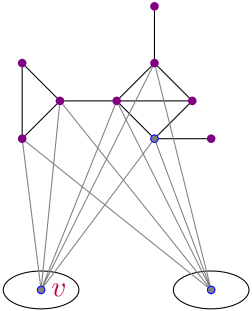


2. if $|F \cap F'| > 1$

add the degree-one vertex of F to M

Modulator

Construction of the modulator M



3. if an isolated vertex v of $G - M$ dominates all edges in G'

Modulator

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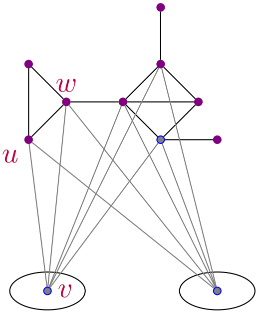


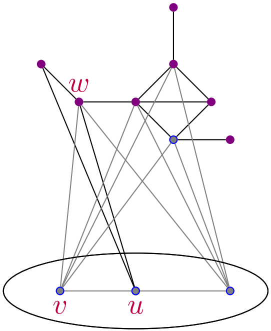
trivial components

3. if an isolated vertex v of $G - M$ dominates all edges in G'

find an edge uw in $G[N(v)]$

Modulator

Construction of the modulator M



type I $\triangle$-free component

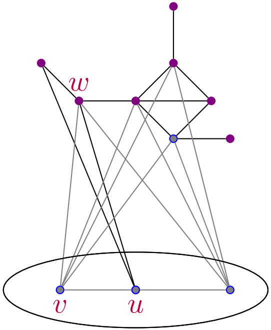
3. if an isolated vertex v of $G - M$ dominates all edges in G'

find an edge uw in $G[N(v)]$

remove u from M

Modulator

Construction of the modulator M



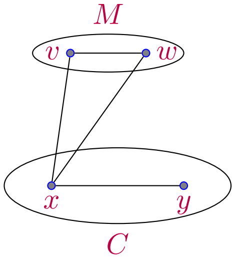
type I $\triangle$-free component

all such trivial components become a type I $\triangle$-free component

Modulator

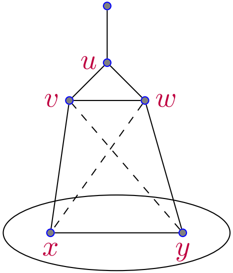
1. M is a modulator
2. $|M| \leq 4k$
3. For each component G' of G, we need to add $\geq |M \cap G'|/4$ edges.
4. All trivial components are considered

Type II triangle-free components



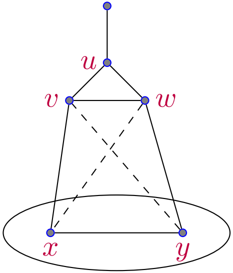
each vertex in a type II $\triangle$-free component of $G - M$ cannot be in a triangle

Type II triangle-free components



each vertex in a type II $\triangle$-free component of $G - M$ is incident to ≥ 1 edge in a solution

Type II triangle-free components



At most k vertices in type II $\triangle$-free components

Then we consider the components of G one by one

Let G' be a component of G and $M' = M \cap V(G')$

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To bound $|V(G') \setminus M'|$

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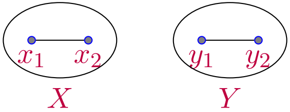
To bound $|V(G') \setminus M'|$

To show the minimum number of edges we need to add to G' is linear on $|V(G') \setminus M'|$

If two components in $G' - M'$ are not type II
$\triangle$-free components



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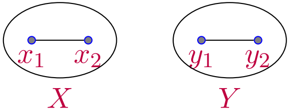


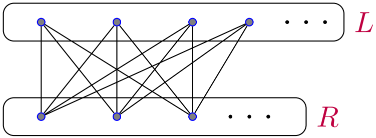
the number of edges we need to add is at least

$$|V(G') \setminus (M' \cup X)| + |V(G') \setminus (M' \cup Y)| - 2 \geq |V(G') \setminus M'|/2$$

$G' - M'$ has precisely one type I $\triangle$-free component or one complete multipartite component

Type I triangle-free component

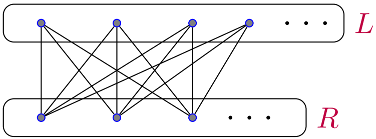
Let C be a type I $\triangle$-free component of $G' - M'$, $L \uplus R$ the bipartition of C with $|L| \geq |R|$



Type I triangle-free component

Lemma

If any of the following conditions is satisfied, then we need to add at least $|C|/32$ edges to G'.



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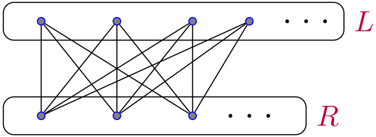
If any of the following conditions is satisfied, then we need to add at least $|C|/32$ edges to G'.

(i) $|L| \leq 4|M'|$;



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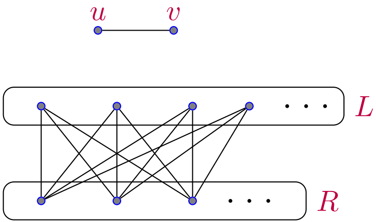

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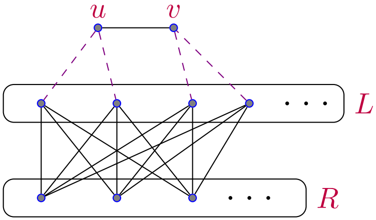

add at least $|L| \geq |C|/2$ edges

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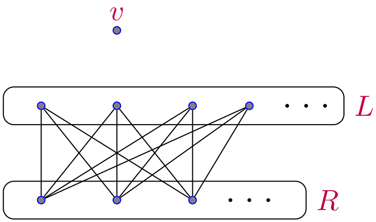
(i) $|L| \leq 4|M'|$;

(ii) there is an edge in $G' - N[L]$;

(iii) $V(G') \neq N[C]$ and $|L| \leq 2|R|$;



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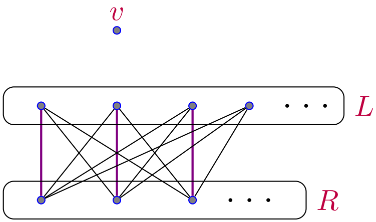

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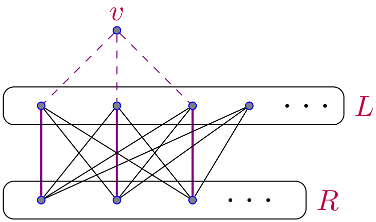


add at least $|R| \geq |C|/3$ edges

Type I triangle-free component

Lemma

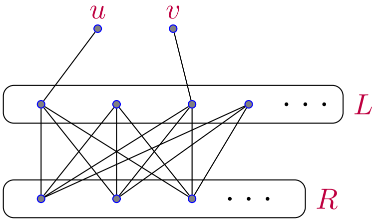
If any of the following conditions is satisfied, then we need to add at least $|C|/32$ edges to G'.

(i) $|L| \leq 4|M'|$;

(ii) there is an edge in $G' - N[L]$;

(iii) $V(G') \neq N[C]$ and $|L| \leq 2|R|$;

(iv) $\geq |L|/2$ missing edges between L and $N(L)$;



add at least $|L|/2 \geq |C|/4$ edges

Type I triangle-free component

Lemma

If any of the following conditions is satisfied, then we need to add at least $|C|/32$ edges to G'.

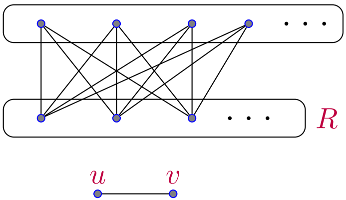
(i) $|L| \le 4|M'|$;

(ii) there is an edge in $G' - N[L]$;

(iii) $V(G') \neq N[C]$ and $|L| \le 2|R|$;

(iv) $\ge |L|/2$ missing edges between L and $N(L)$;

(v) $|L| \le |R| + |M'|$ and $G - N[R]$ has an edge;



add at least $|C|/3$ edges

Type I triangle-free component

Lemma

If any of the following conditions is satisfied, then we need to add at least $|C|/32$ edges to G'.

(i) $|L| \leq 4|M'|$;

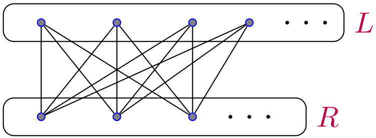
(ii) there is an edge in $G' - N[L]$;

(iii) $V(G') \neq N[C]$ and $|L| \leq 2|R|$;

(iv) $\geq |L|/2$ missing edges between L and $N(L)$;

(v) $|L| \leq |R| + |M'|$ and $G - N[R]$ has an edge;

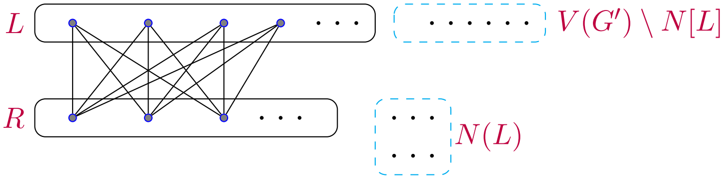
(vi) $|L| \leq |R| + |M'|$ and $\geq |R|/2$ missing edges between R and $N(R)$



add at least $|C|/6$ edges

Type I triangle-free component

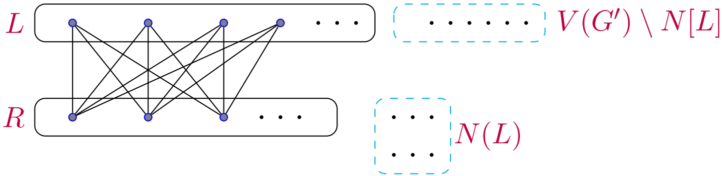
Rule. If none of the conditions holds true,



$V(G') \setminus N[L]$ is an independent set

Type I triangle-free component

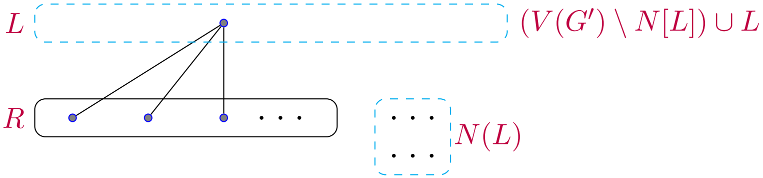
Rule. If none of the conditions holds true,



add all the missing edges between L and $N(L)$;
add all the missing edges between $V(G') \setminus N[L]$ and $N(L)$;

Type I triangle-free component

Rule. If none of the conditions holds true,



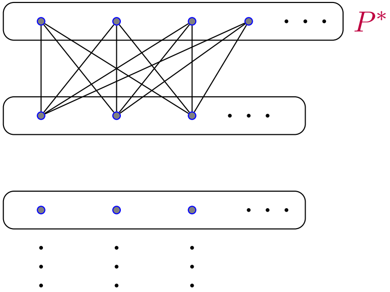
add all the missing edges between L and $N(L)$;
add all the missing edges between $V(G') \setminus N[L]$ and $N(L)$;
remove all but one vertex from $(V(G') \setminus N[L]) \cup L$.

Type I triangle-free component

at most $32k$ vertices in type I $\triangle$-free components of $G - M$

Complete multipartite component

Let C be a complete multipartite component of $G' - M'$, P^* a largest part of C.



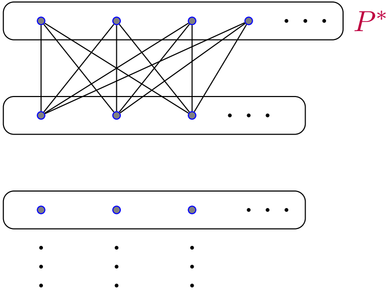
Complete multipartite component

Lemma

If any of the following conditions is satisfied, then we need to add at least $|C|/12$ edges to G'.

(i) $|C| \leq 3|M'|$;

(ii) there is an edge in $G' - N[C]$;



add at least $|C|$ edges

Complete multipartite component

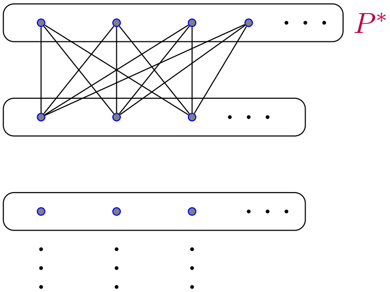
Lemma

If any of the following conditions is satisfied, then we need to add at least $|C|/12$ edges to G'.

(i) $|C| \leq 3|M'|$;

(ii) there is an edge in $G' - N[C]$;

(iii) $|P^*| > 2|C|/3$ and $G' - N[P^*]$ has an edge;



add at least $2|C|/3$ edges

Complete multipartite component

Lemma

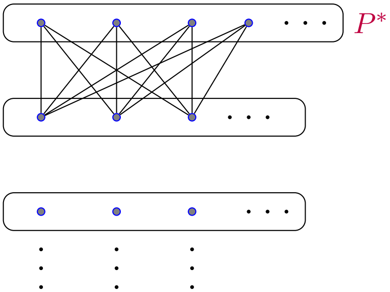
If any of the following conditions is satisfied, then we need to add at least $|C|/12$ edges to G'.

(i) $|C| \leq 3|M'|$;

(ii) there is an edge in $G' - N[C]$;

(iii) $|P^*| > 2|C|/3$ and $G' - N[P^*]$ has an edge;

(iv) $|P^*| \leq 2|C|/3$ and $V(G') \neq N[C]$;



add at least $|C|/3$ edges

Complete multipartite component

Lemma

If any of the following conditions is satisfied, then we need to add at least $|C|/12$ edges to G'.

(i) $|C| \leq 3|M'|$;

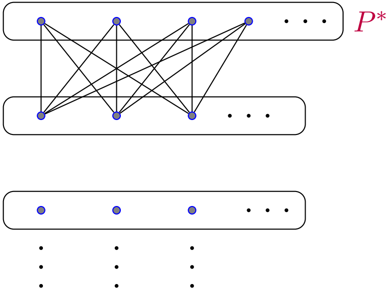
(ii) there is an edge in $G' - N[C]$;

(iii) $|P^*| > 2|C|/3$ and $G' - N[P^*]$ has an edge;

(iv) $|P^*| \leq 2|C|/3$ and $V(G') \neq N[C]$;

(v) $|P^*| \leq 2|C|/3$ and $V(G') = N[C]$, for every P,

1. $G' - N[P]$ contains an edge, or
2. $\geq |P|$ missing edges between $V(G') \setminus N[P]$ and $N(P)$.



Complete multipartite component

Rule. If none of the conditions holds true, then

1. if $|P^*| > 2|C|/3$, add missing edges between $V(G') \setminus N[P^*]$ and $N(P^*)$, remove $(V(G') \setminus N[P^*]) \cup P^*$;

2. find a P, add missing edges between $V(G') \setminus N[P]$ and $N(P)$, remove $(V(G') \setminus N[P]) \cup P$

Complete multipartite component

at most $12k$ vertices in complete multipartite components of $G - M$

Theorem

The paw-free completion problem has a $38k$-vertex kernel.

Thanks!