Structured Connectivity Augmentation

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Connectivity augmentation

- András Frank. Connections in combinatorial optimization. Volume 38 of Oxford Lecture Series in Mathematics and its Applications. Oxford University Press, Oxford, 2011.
- Hiroshi Nagamochi and Toshihide Ibaraki. Algorithmic aspects of graph connectivity. Volume 123 of Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 2008.

Graph superposition

Let G and H be simple graphs, $|V(G)| \geq |V(H)|$, and let

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- $uv \in E(G)$ or
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We write $F = G \oplus_\varphi H$.

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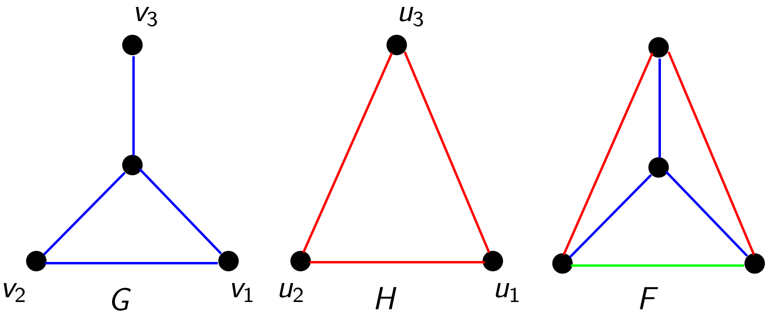

$F = G \oplus_\varphi H$ where $\varphi(u_i) = v_i$ for $i = 1, 2, 3$.

Structured Connectivity Augmentation

Let k be a positive integer (we are mainly interested in the cases $k = 1, 2$).

Structured k-Connectivity Augmentation

Input: Graphs G and H such that G is edge $(k-1)$-connected, a weight function $\omega\colon \binom{V(G)}{2} \to \mathbb{N}_0$ and a nonnegative integer W.

Task: Decide whether there is an injective $\varphi\colon V(H) \to V(G)$ such that $F = G \oplus_{\varphi} H$ is edge k-connected and the weight of the mapping $\omega(\varphi) = \sum_{xy \in E(H)} \omega(\varphi(x)\varphi(y)) \leq W$.

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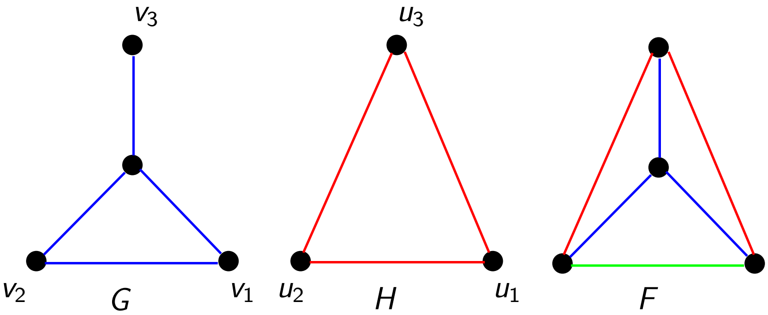
For $k = 1$, we call the problem Structured Connectivity Augmentation.

Structured Connectivity Augmentation



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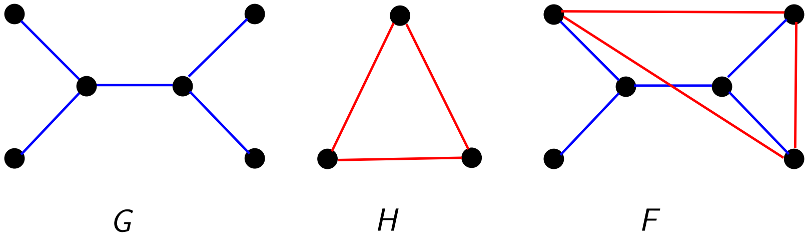

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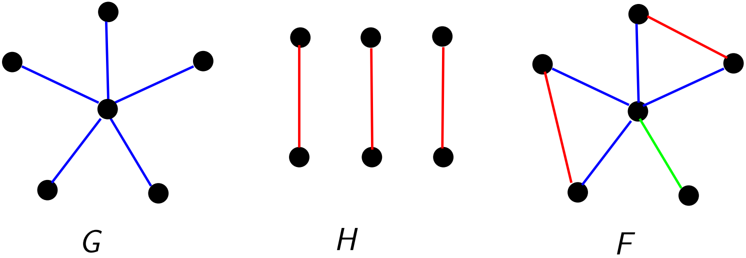

Our results

We say that a class of graphs $\mathcal{C}$ has bounded vertex-cover number, if there is a constant t depending on $\mathcal{C}$ only such that the vertex-cover number of every graph from $\mathcal{C}$ does not exceed t.

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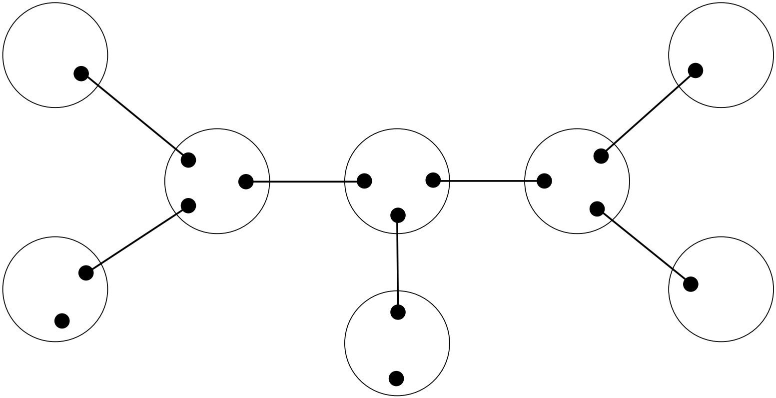
- For unweighted case, we obtain necessary and sufficient combinatorial conditions of the existence of an injective function φ such that $F = G \oplus_\varphi H$ is edge k-connected provided that G is edge $(k-1)$-connected for $k = 1, 2$.

Structured 2-Connectivity Augmentation

Theorem

Let t be a positive integer and $\mathcal{C}$ be a graph class of vertex-cover number at most t. Then for any $H \in \mathcal{C}$, Structured 2-Connectivity Augmentation is solvable in time $|V(G)|^{\mathcal{O}(2^t)} \log W$.

Covering bridges



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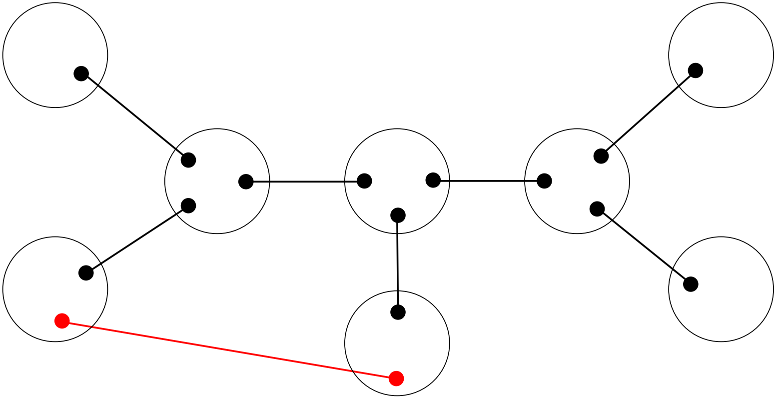
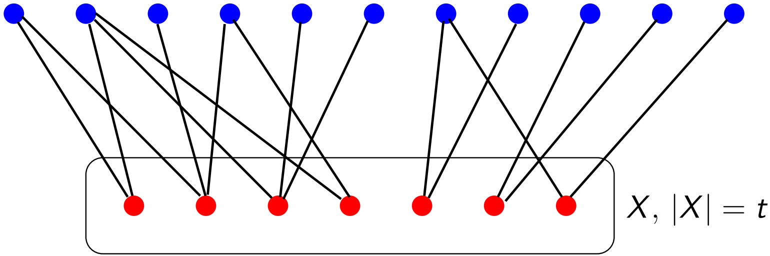
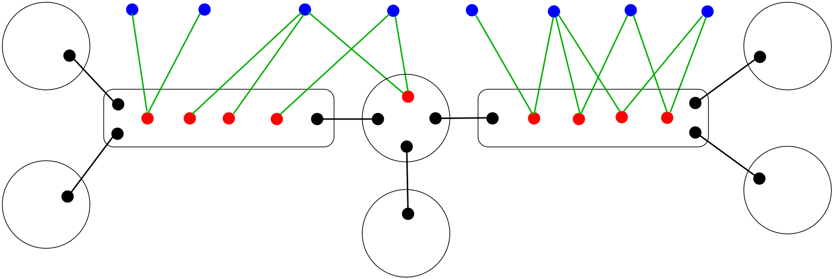


Graphs of bounded vertex-cover number



Partial mapping



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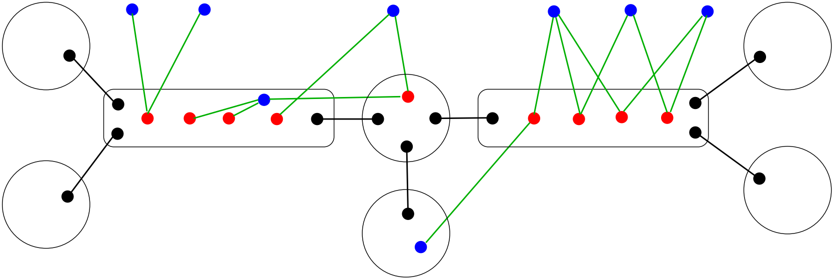
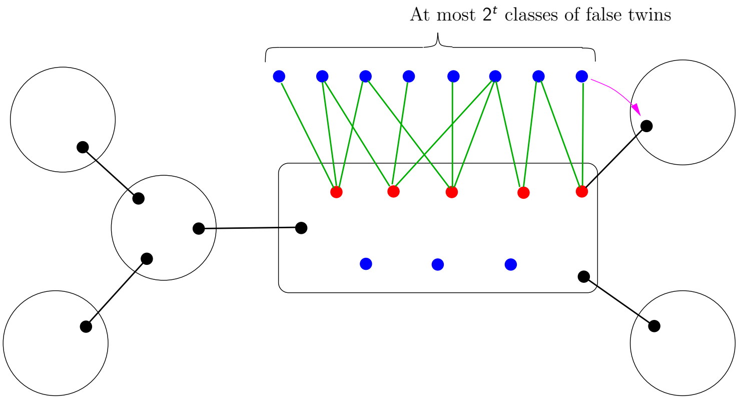


Partial mapping

Lemma

Let G and H be graphs such that G is connected, and let $\varphi \colon V(H) \to V(G)$ be an injection such that $F = G \oplus_\varphi H$ is 2-connected. Suppose that X is a vertex cover of H and $t = |X|$. Then there is a set $Y \subseteq V(H) \setminus X$ of size at most $2(t-1)$ such that for $H' = H[X \cup Y]$ and $\psi = \varphi|_{X \cup Y}$, the vertices of $\psi(X \cup Y)$ are in the same biconnected component of $F' = G \oplus_\psi H'$.

Extension of a partial mapping



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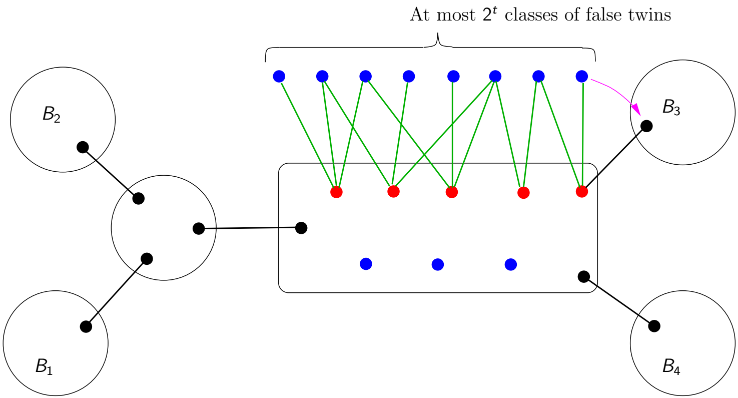

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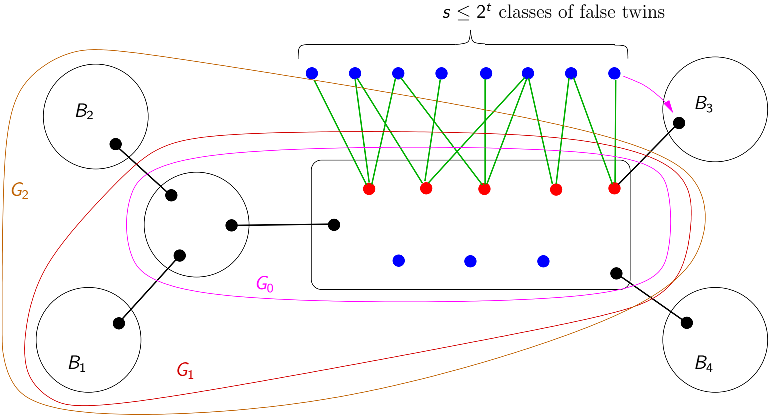

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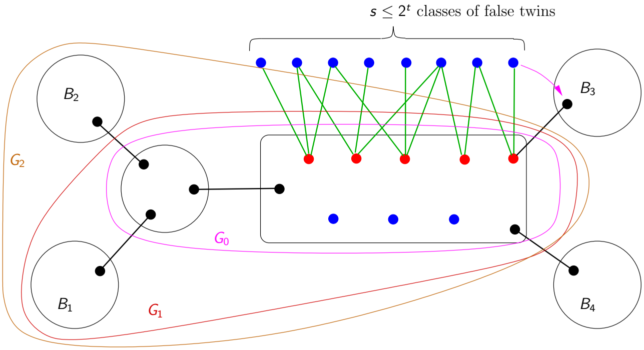

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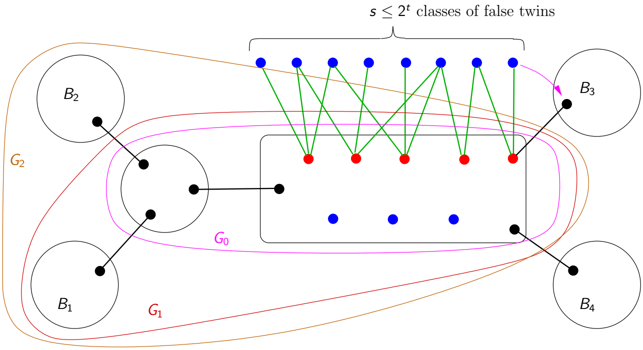
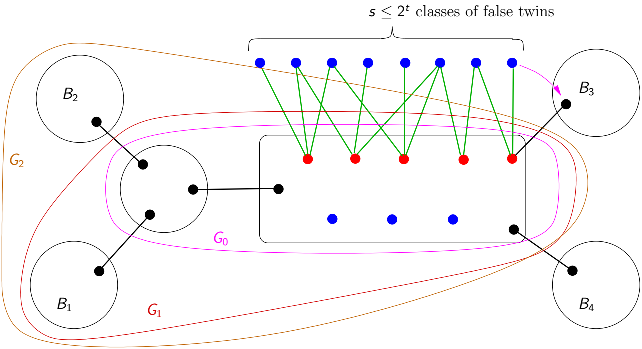


For $(\ell_1, \ldots, \ell_s)$, find an injective mapping of minimum weight of a subset of the vertices of H containing ℓ_j vertices of the j-th class of true twins for $j \in \{1, \ldots, s\}$ to G_i such that some vertices are mapped to $B_1, \ldots, B_i$ to ensure that the corresponding bridges are covered.

Extension of a partial mapping



Consider all sums $(\ell_1, \ldots, \ell_s) = (\ell'_1, \ldots, \ell'_s) + (\ell''_1, \ldots, \ell''_s)$ and find optimal mapping where for $j \in \{1, \ldots, s\}$, ℓ'_j vertices from the j-th class are mapped to B_i and ℓ''_j vertces are mapped to G_{i-1}.

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Hardness

Theorem

Let k be a positive integer. Let also $\mathcal{C}$ be a hereditary graph class. Then if the vertex-cover number of $\mathcal{C}$ is unbounded, then Structured k-Connectivity Augmentation is NP*-complete for $H \in \mathcal{C}$ in the strong sense.*

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- $K_{n,n} \in \mathcal{C}$ for all $n \in \mathbb{N}$,
- the matching graph $M_n \in \mathbb{N}$ (disjoint union of n copies of K_2) for all $n \in \mathbb{N}$.

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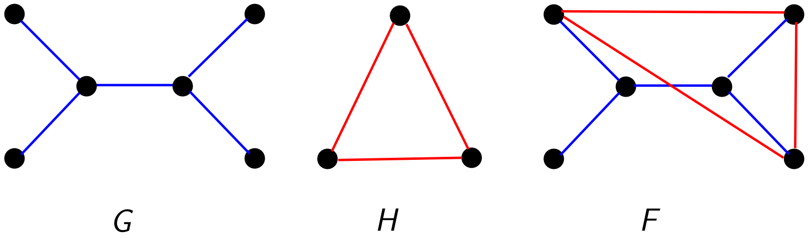
Proposition

For every positive integer k, Structured k-Connectivity Augmentation is W[1]*-hard when parameterized by $\beta(H)$ even if the weight of every pair of vertices of G is restricted to be ether* 0 *or* 1.

Unweighted augmentation



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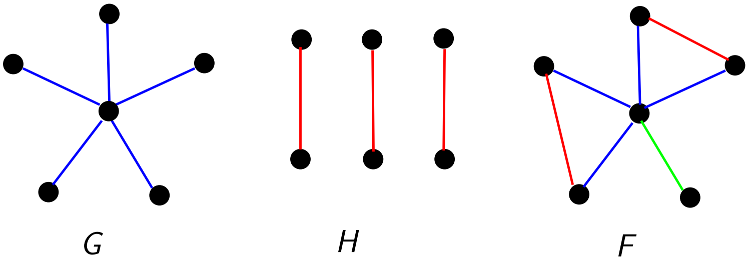

Unweighted augmentation

Theorem

Let G and H be graphs such that G is connected, H has no isolated vertices and $|V(H)| \leq |V(G)|$. Then there is an injective mapping $\varphi\colon V(H) \to V(G)$ such that $F = G \oplus_\varphi H$ is 2-connected if and only if one of the following holds:

(i) *G is 2-connected,*

(ii) *G is not 2-connected and $p(G) \leq |V(H)|$ where $p(G)$ is the number of pendant biconnected components.*

unless G is a star $K_{1,n}$ where n is odd and H is a matching graph.

Conclusion

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Open questions

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- What can be said about the variant of Structured k-Connectivity Augmentation where the aim is to increase the vertex connectivity?

Thank You!