A Polynomial Kernel for Funnel Arc Deletion Set

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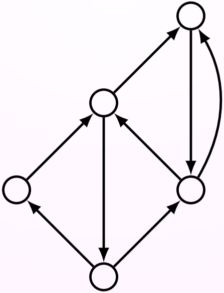
Contents

Let D be a digraph and $k \in \mathbb{N}$.

DIRECTED FEEDBACK ARC SET (DFAS)

Is there some $A' \subseteq A(D)$ such that $D - A'$ is a directed acyclic graph (DAG) and $|A'| \leq k$?



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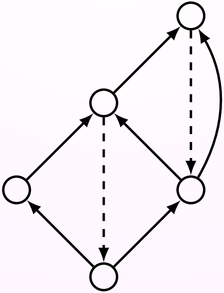

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$\mathcal{C}$-Arc Deletion Set ($\mathcal{C}$-ADS)

Is there some $A' \subseteq A(D)$ such that $D - A' \in \mathcal{C}$ and $|A'| \leq k$?

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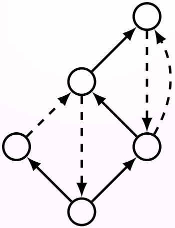

Example for Out-Forest-ADS.

Definition (Funnel)

A DAG D is a funnel if for every source-sink path P there is some arc $a \in A(P)$ such that $Q = P$ for any source-sink path Q containing a.

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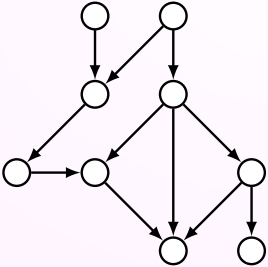

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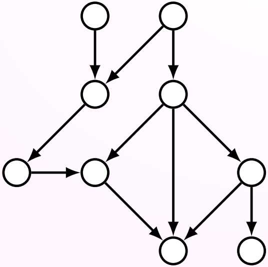
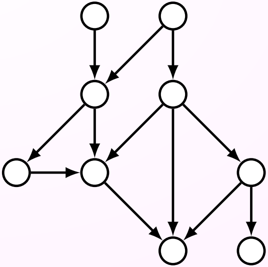
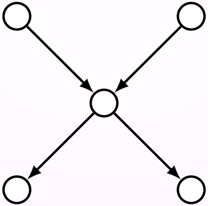


Not a funnel



Forbidden Butterfly-Minor

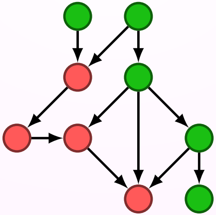
A DAG is a funnel if and only if it does not contain the following as a butterfly minor.



Partitioning

A DAG D is a funnel if and only if $V(D) = \mathrm{F} \uplus \mathrm{M}$ such that

1. $D[\mathrm{F}]$ is an out-forest,
2. $D[\mathrm{M}]$ is an in-forest, and
3. $\mathrm{M} \times \mathrm{F} \cap A(D) = \emptyset$.



- There is a $\mathcal{O}(4^k k^4 nm)$ time algorithm for DFAS/DFVS.[1]
- DFVS admits a polynomial kernel when parameterized by k and the size of a treewidth-η modulator (for constant η).[2]

[1] J. Chen, Y. Liu, S. Lu, B. O'Sullivan., I. Razgon. *A fixed-parameter algorithm for the directed feedback vertex set problem*. 2008.

[2] D. Lokshtanov, M. S. Ramanujan, S. Saurabh, R. Sharma, M. Zehavi. *Wannabe bounded treewidth graphs admit a polynomial kernel for DFVS*. 2019.

- Out-Forest-ADS and Pumpkin-ADS can be solved in polynomial time.[3]
- Out-Forest-VDS and Pumpkin-VDS are NP-hard and admit polynomial kernels wrt. solution size.[3]

[3] M. Mnich and E. J. van Leeuwen. *Polynomial kernels for deletion to classes of acyclic digraphs*. 2017.

[4] F. V. Fomin, P. A. Golovach, D. M. Thilikos. *On the Parameterized Complexity of Graph Modification to First-Order Logic Properties*. 2020

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- OUT-FOREST-VDS and PUMPKIN-VDS are NP-hard and admit polynomial kernels wrt. solution size.[3]
- $\mathcal{C}$-ADS admits a polynomial kernel if $\mathcal{C}$ "can be described by a simple first-order logic formula".[4]

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- Funnel-ADS is NP-hard even if the input is a DAG.[5]
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- Compare: OUT-FOREST-ADS and PUMPKIN-ADS can be solved in polynomial time.

Theorem

FUNNEL-ADS admits a kernel with $\mathcal{O}(k^6)$ many vertices and $\mathcal{O}(k^7)$ many arcs, computable in $\mathcal{O}(n+m)$ time.

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Contents

Let D be a digraph, $\ell : V(D) \to \{\mathrm{F}, \mathrm{M}\}$ a (partial) labeling of the vertices and $k \in \mathbb{N}$.

FUNNEL ARC DELETION LABELING (FADL)

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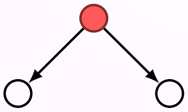

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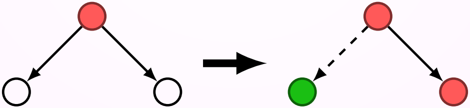
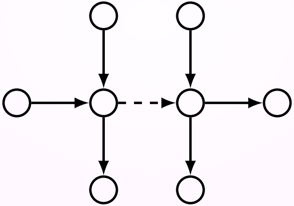


Reduction Rule

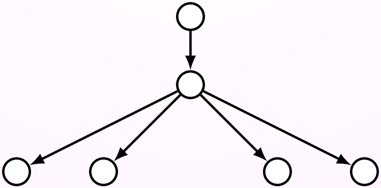
If there are more than $2k$ vertices with both in- and outdegree greater than 1, reject the input.



Basic Reduction Rules

Reduction Rule

1. Label with F every source vertex.
2. Label with M every sink vertex.



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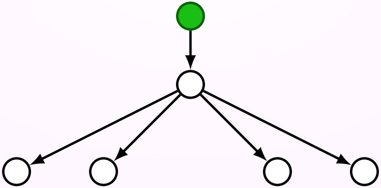

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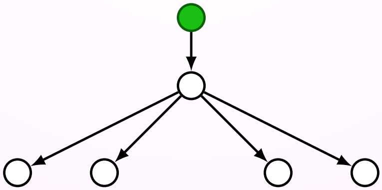

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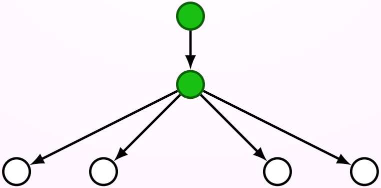

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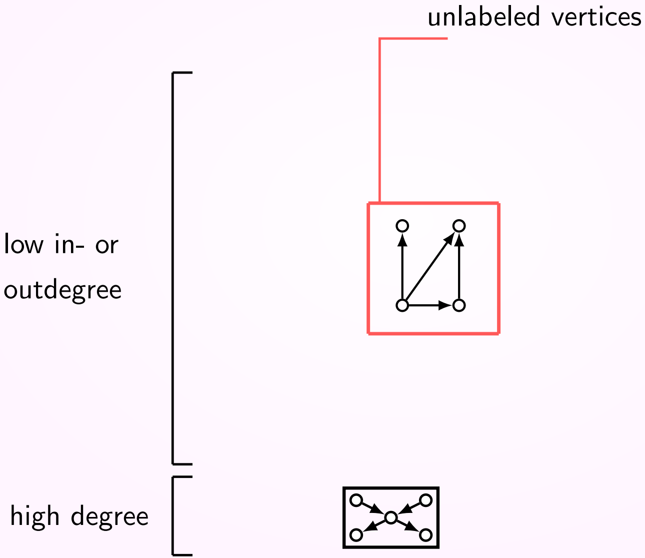

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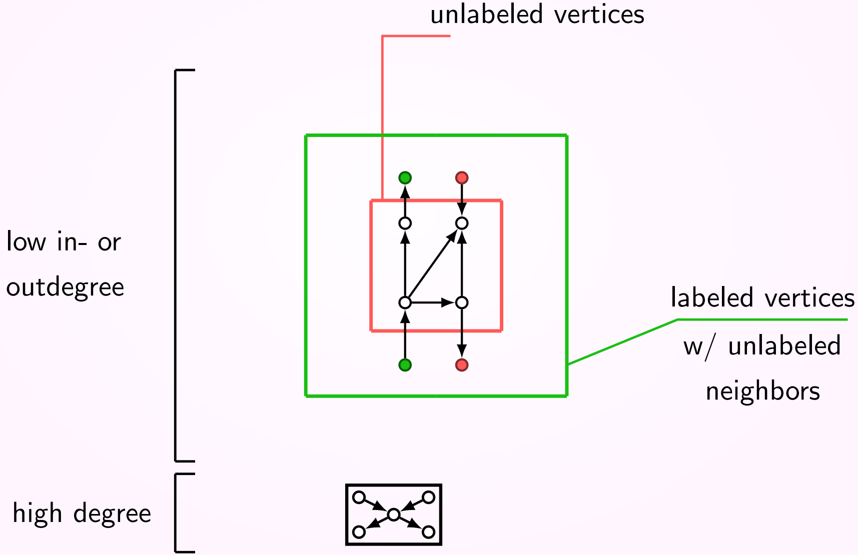

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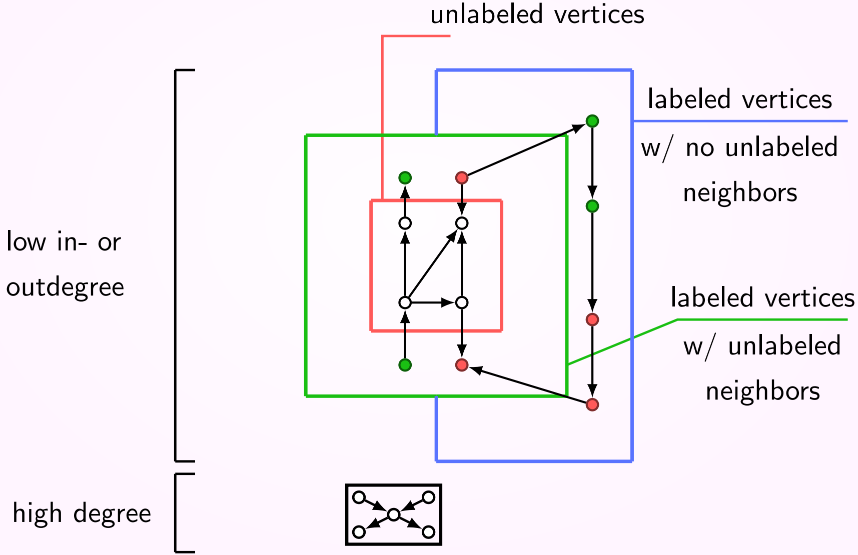
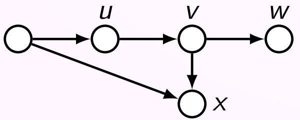


Reduction Rule (shift neighbors)

Let u, v, w be a path of unlabeled vertices such that all of them have indegree one.
We can "shift" a neighbor x of v "backwards" if it is not a neighbor of u.



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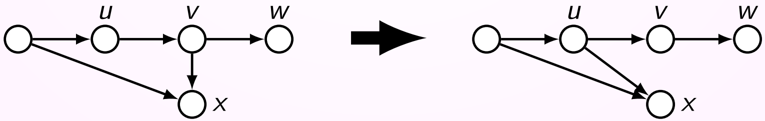
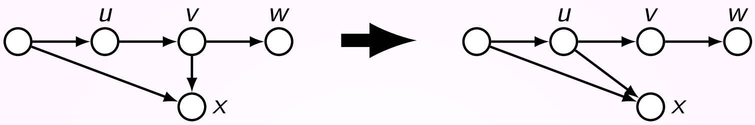


Unlabeled Vertices w/ Low Degree



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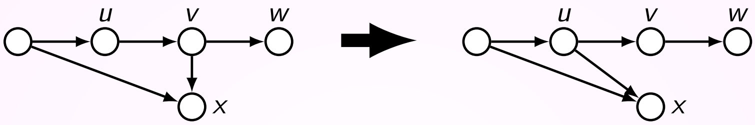

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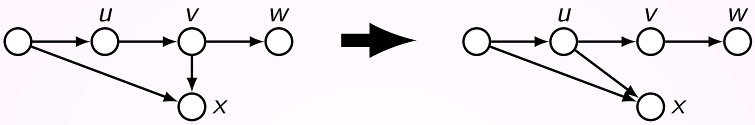

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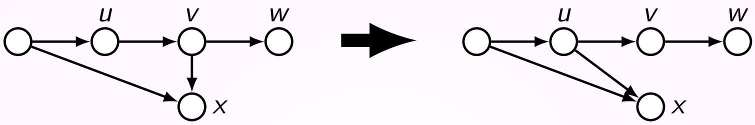

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4. P is short.

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With an additional reduction rule, we can prove the following.

Lemma

Let D be a reduced digraph. Then there are $\mathcal{O}(k^5)$ unlabeled vertices with in- or outdegree one.

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Corollary

There are $\mathcal{O}(k^6)$ labeled vertices with unlabeled neighbors.

Labeled Vertices w/ Labeled Neighbors

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Remove (v, u) if v, u have different labels. Further, set $k := k - 1$ if v is labeled with M and u is labeled with F.

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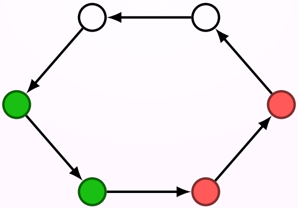

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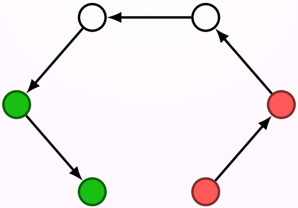

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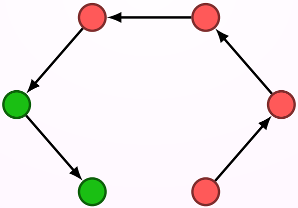

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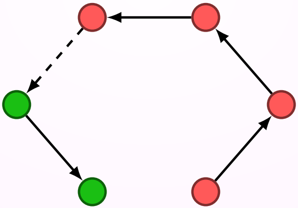

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Putting together previous statements :

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And with a simple reduction from FADL to FADS (and with a running-time analysis), we obtain:

Theorem

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Thank you.