Subgraph Complementation

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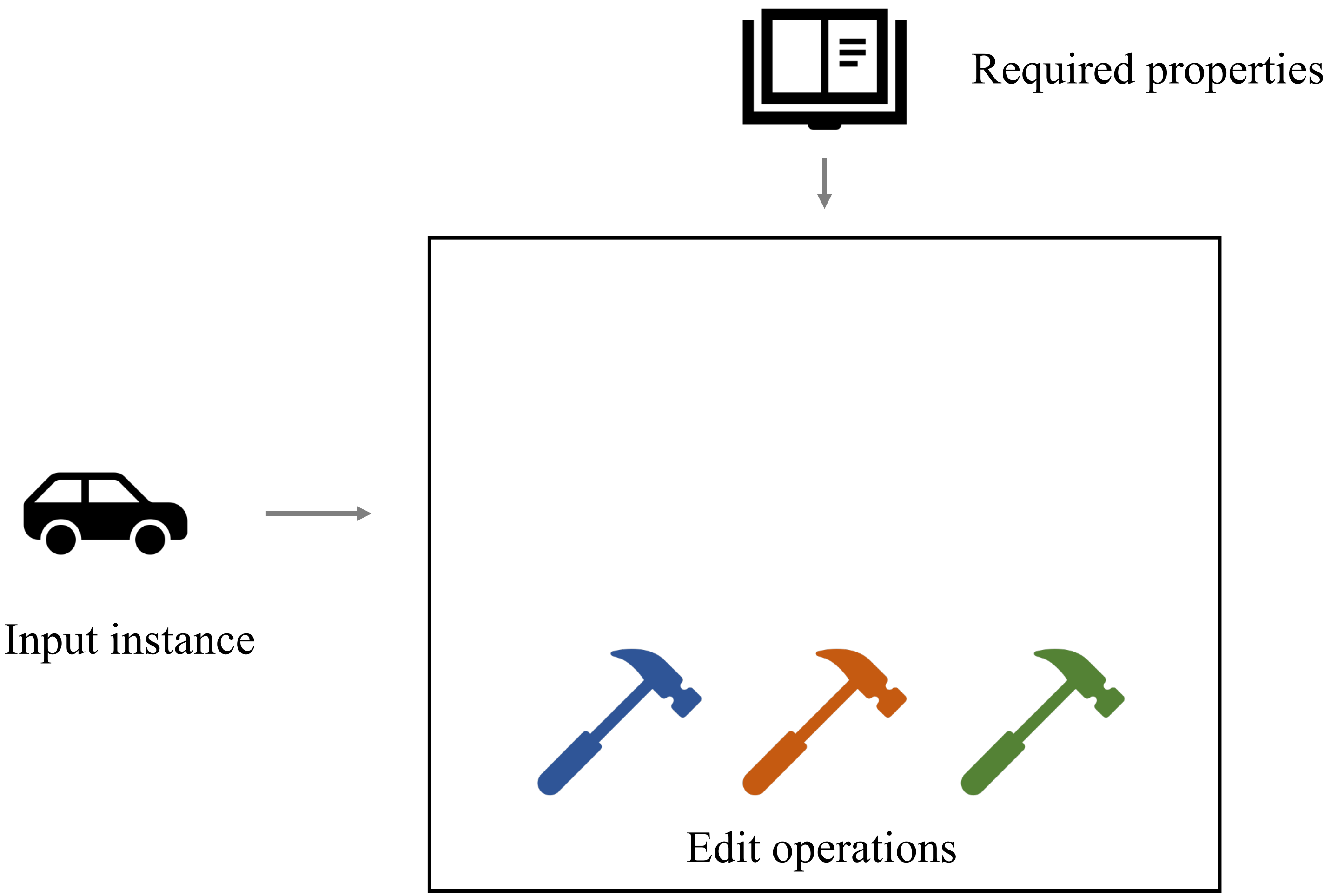
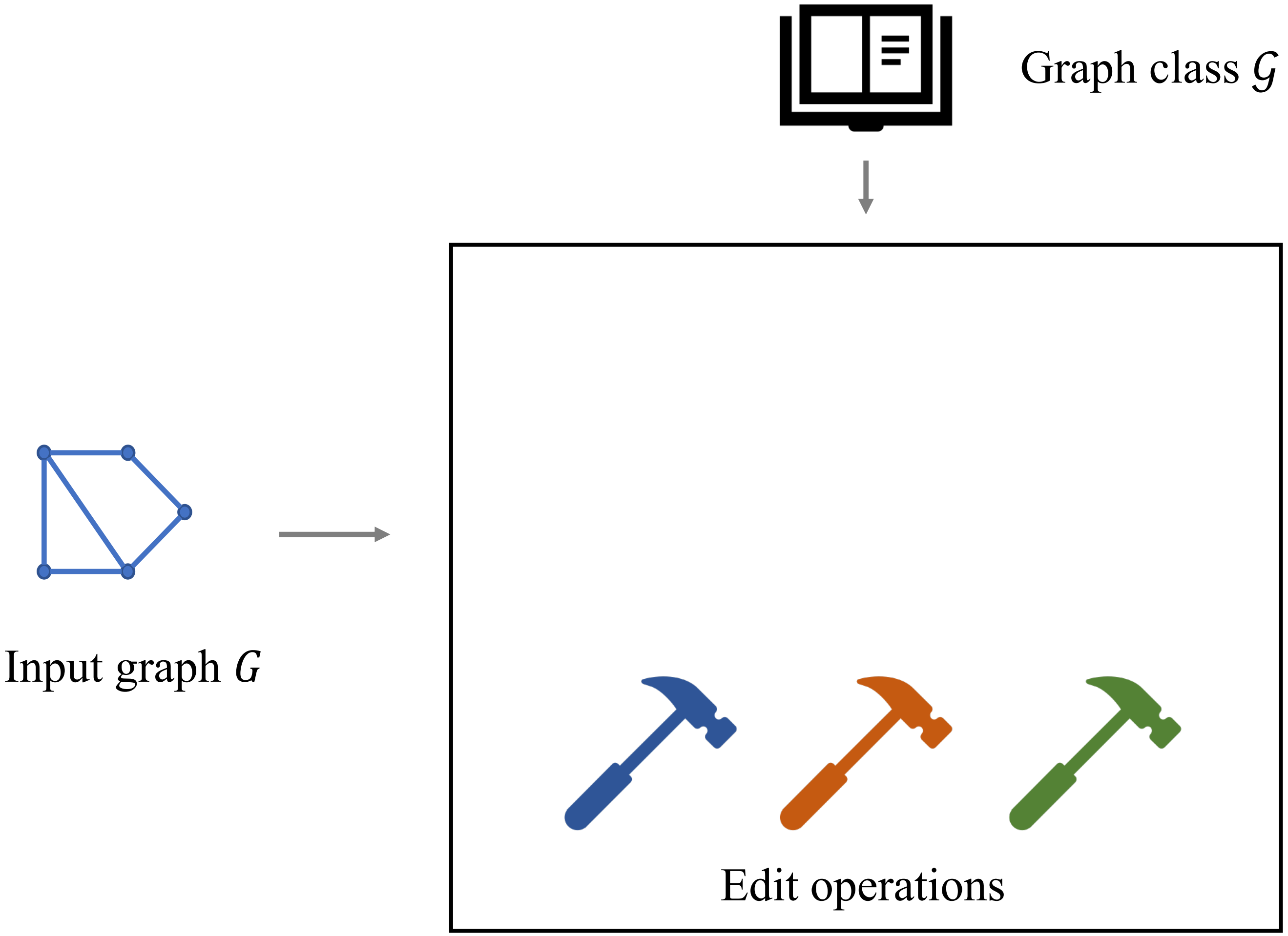
Editing



Question:

Can I do at most k edit operations to my instance to make it satisfy the required properties?

Graph editing



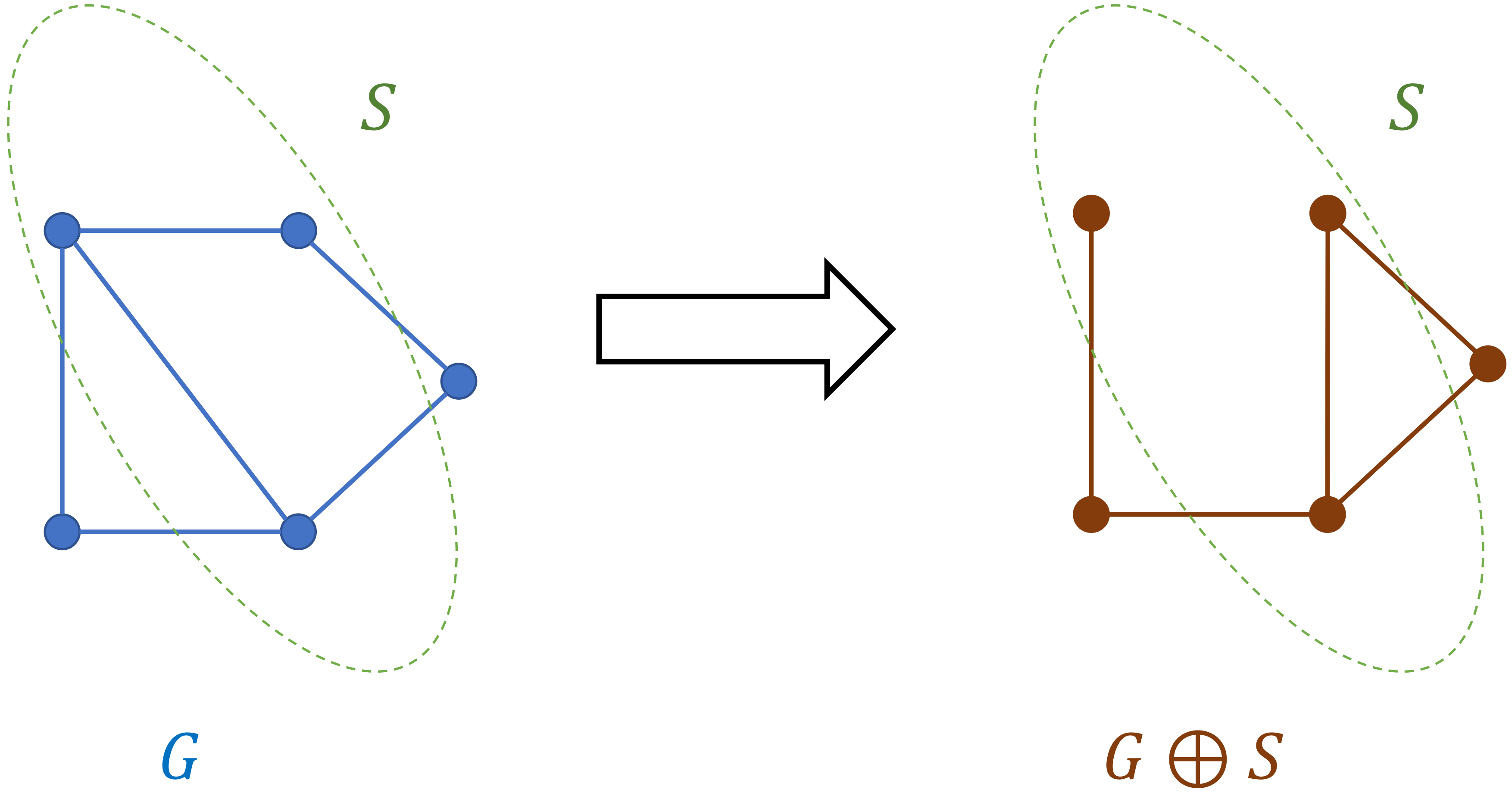
Question:

Can I do at most k edit operations to G to find a member of $\mathcal{G}$?

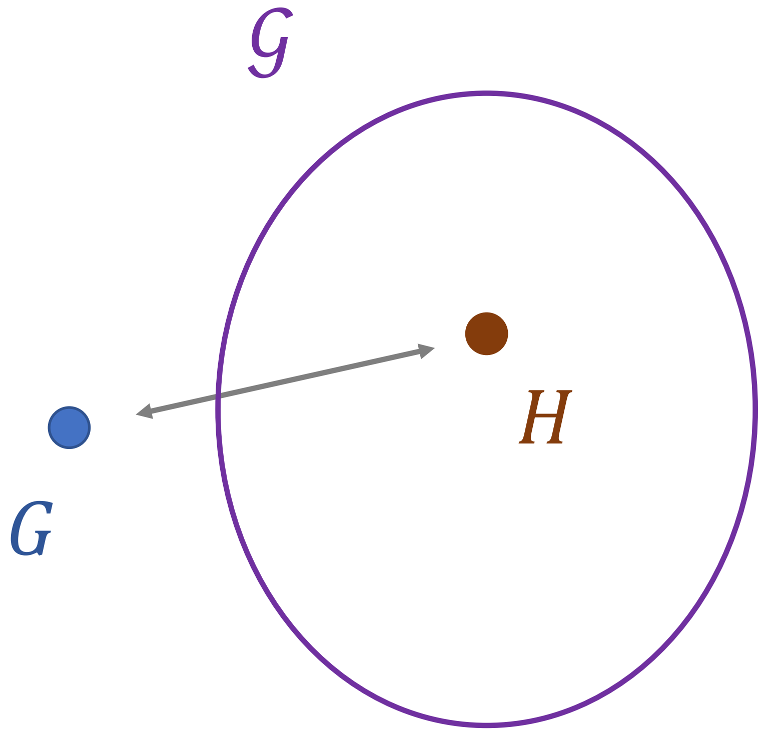
Graph editing

Edit operations	**Graph class $\mathcal{G}$**	$\boldsymbol{k}$	**Problem**
Vertex deletion	Edgeless	k	Vertex Cover
Vertex deletion	Union of trees	k	Feedback Vertex Set
Edge insertion	Chordal	k	Minimum Fill-in
Edge insertion, edge deletion	Union of cliques	k	Cluster Editing
Vertex deletion, edge deletion, edge contraction	$\{H\}$	∞	H-minor
Vertex deletion, edge contraction	$\{H\}$	∞	H-topological-minor
Seidel switch	$\mathcal{G}$	1	
Subgraph complement	$\mathcal{G}$	1	

Subgraph complementation



Subgraph complementation



Graph G can be
partially complemented
to graph class $\mathcal{G}$

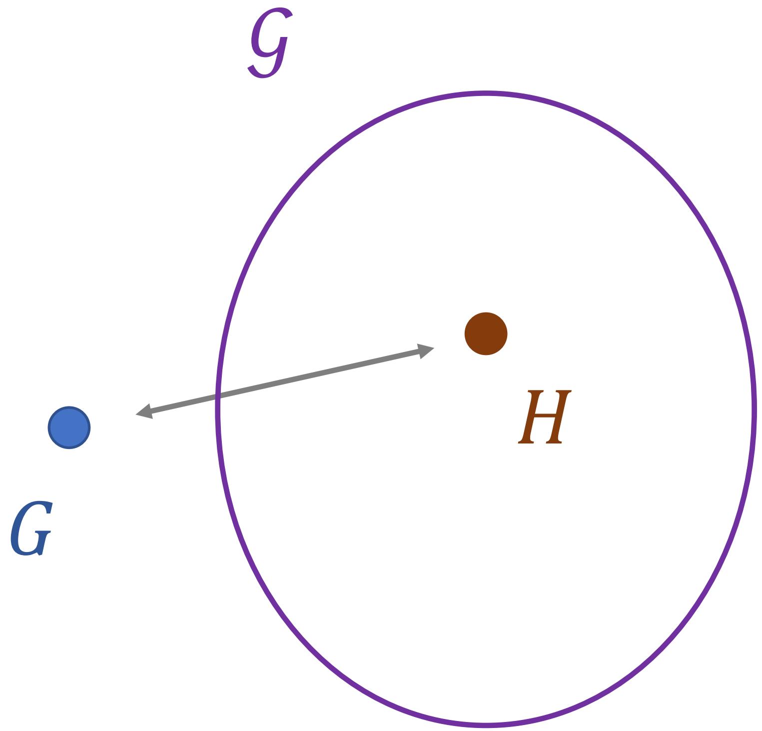
$\Updownarrow$

$\exists\, H \in \mathcal{G}$
$\exists\, S \subseteq V(G)$

such that

$G \oplus S \cong H$

Subgraph complementation



Input:

Graph G, graph class $\mathcal{G}$

Question:

Can G be subgraph complemented to $\mathcal{G}$?

Introduced by [Kaminski, Lozin, Milanic 2009]

Our results

This talk

Graph class $\mathcal{G}$	**Complexity**
All triangle-free classes*	P
All d-degenerate graph classes*	P
Classes of bounded clique-width expressible in MSO_2	P
Graph classes which can be described by $2{\times}2$ M-partition (e.g. split graphs)	P
Regular graphs	NP -complete

* which are checkable in polynomial time

Triangle-free

Theorem
Subgraph complementation to a triangle-free graph class $\mathcal{G}$ is solvable in polynomial time if the class $\mathcal{G}$ can be recognized in polynomial time.

Proof:
By algorithm.

Triangle-free

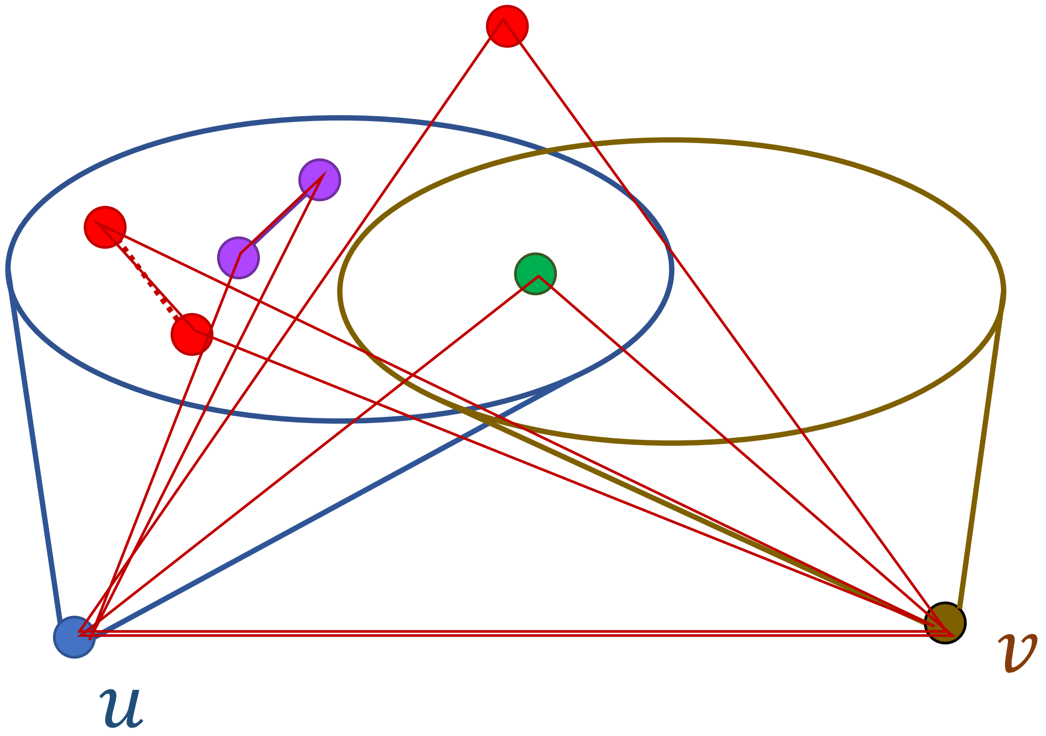
Algorithm:

1. Check if there exists a solution of size ≤ 2
2. Check if there exists a solution containing a non-edge in G
3. Check if there exists a solution which is a clique in G
4. If none of the above, no solution

Triangle-free

1. Check if there is solution ≤ 2
2. Check for a solution with non-edge
3. Check for a solution which is a clique
4. If none of the above, no solution

1. Guess a non-edge uv



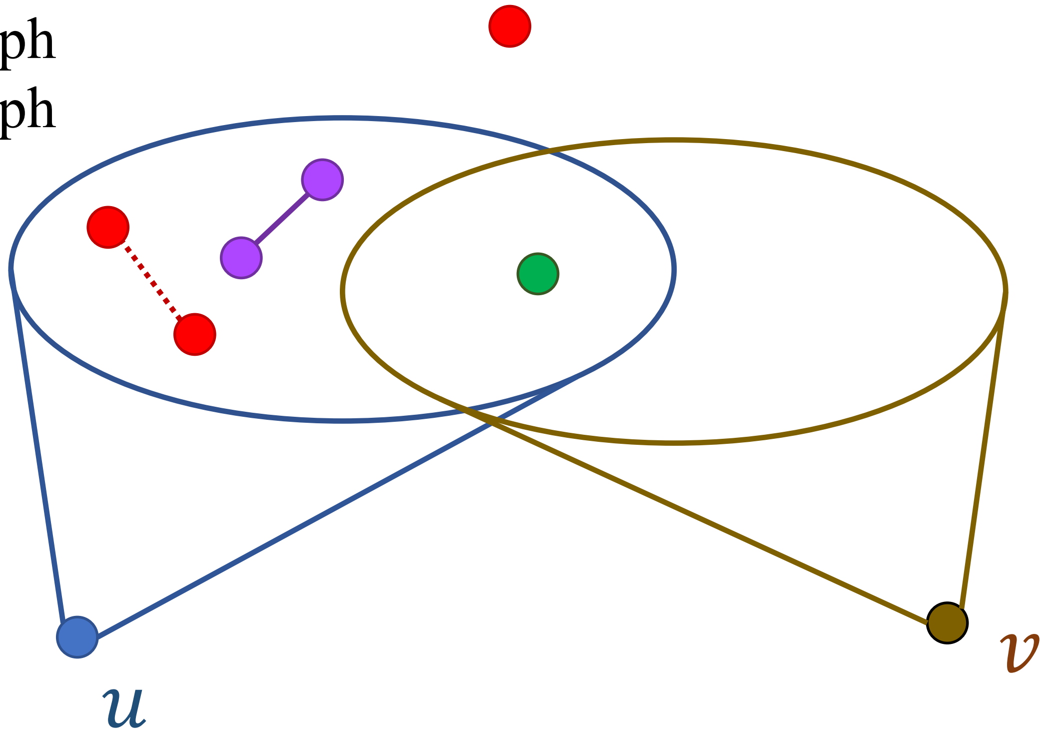
- Solution is $\subseteq N[u] \cup N[v]$
- $N(u) \cap N(v) \subseteq$ solution
- Solution $\cap (N(u) \setminus N(v))$ is a clique
- $\big(N(u) \setminus N(v)\big) \setminus$ solution is independent

Triangle-free

1. Check if there is solution ≤ 2
2. Check for a solution with non-edge
3. Check for a solution which is a clique
4. If none of the above, no solution

1. Guess a non-edge uv
2. Confirm that $N(u) \setminus N(v)$ is split graph
3. Confirm that $N(v) \setminus N(u)$ is split graph
4. Guess which split partitions to use

- Solution is $\subseteq N[u] \cup N[v]$
- $N(u) \cap N(v) \subseteq$ solution
- Solution $\cap (N(u) \setminus N(v))$ is a clique
- $\left(N(u) \setminus N(v)\right) \setminus$ solution is independent

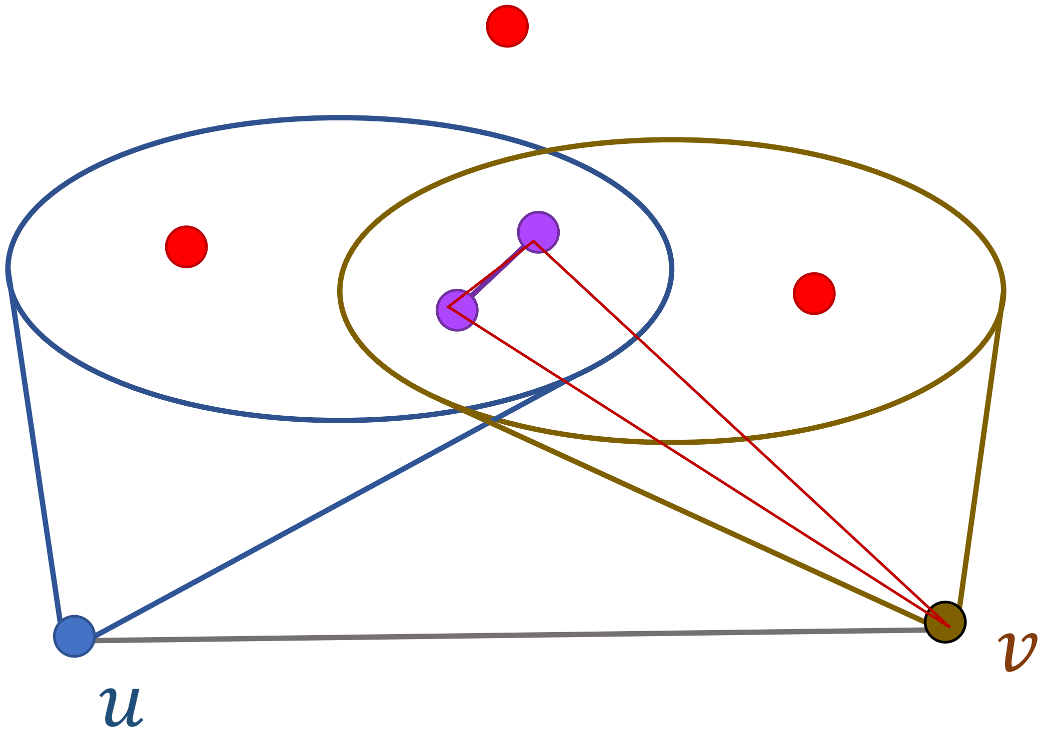


Triangle-free

1. Check if there is solution ≤ 2
2. Check for a solution with non-edge
3. Check for a solution which is a clique
4. If none of the above, no solution

1. Pick any triangle
2. Guess which two are in the solution

- Solution is $\subseteq N[u] \cap N[v]$
- Solution $\cap N(u) \cap N(v)$ is a clique
- $\big(N(u) \cap N(v)\big) \setminus$ solution is independent



Triangle-free

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1. Pick any triangle
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3. Confirm that $N(u) \cap N(v)$ is a split graph
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- Solution is $\subseteq N[u] \cap N[v]$
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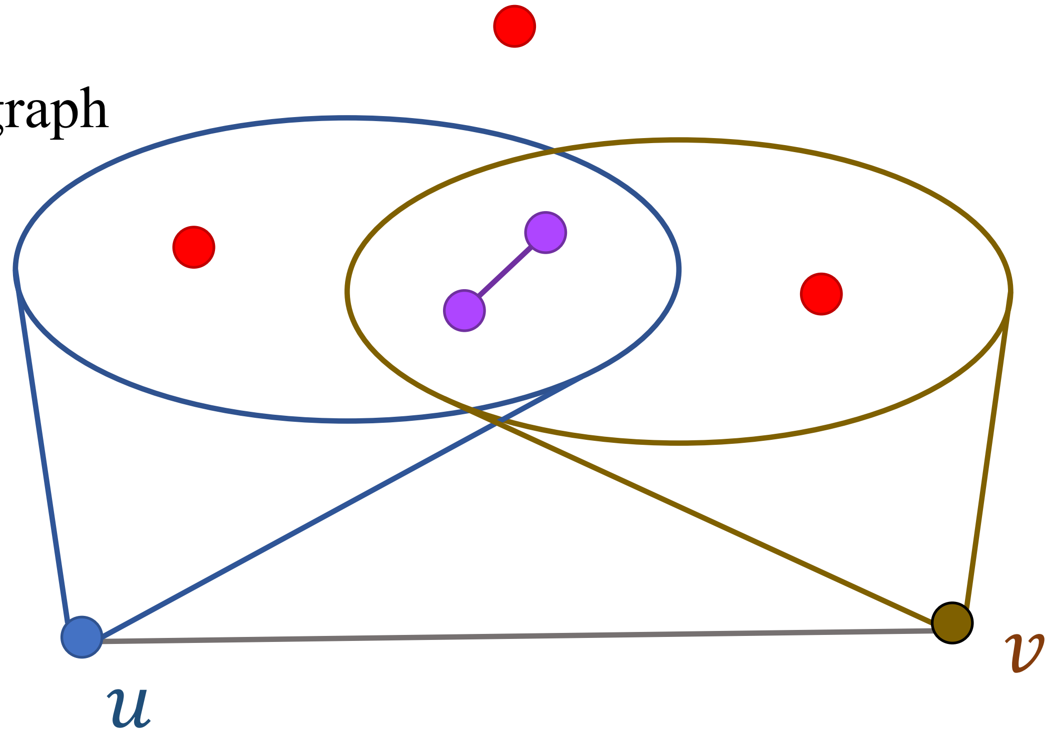


Triangle-free

Algorithm:

1. Check if there exists a solution of size ≤ 2
2. Check if there exists a solution containing a non-edge in G
3. Check if there exists a solution which is a clique in G
4. If none of the above, no solution

Runtime: $\mathcal{O}(f(n) \cdot n^4 + n^6)$

d-degenerate

Graph is d-degenerate
$\Updownarrow$
Every subgraph contains
a vertex of degree $\leq d$

Theorem
Subgraph complementation to a d-degenerate graph class $\mathcal{G}$ is solvable in polynomial time if the class $\mathcal{G}$ can be recognized in polynomial time.

Proof:
By algorithm.

d-degenerate

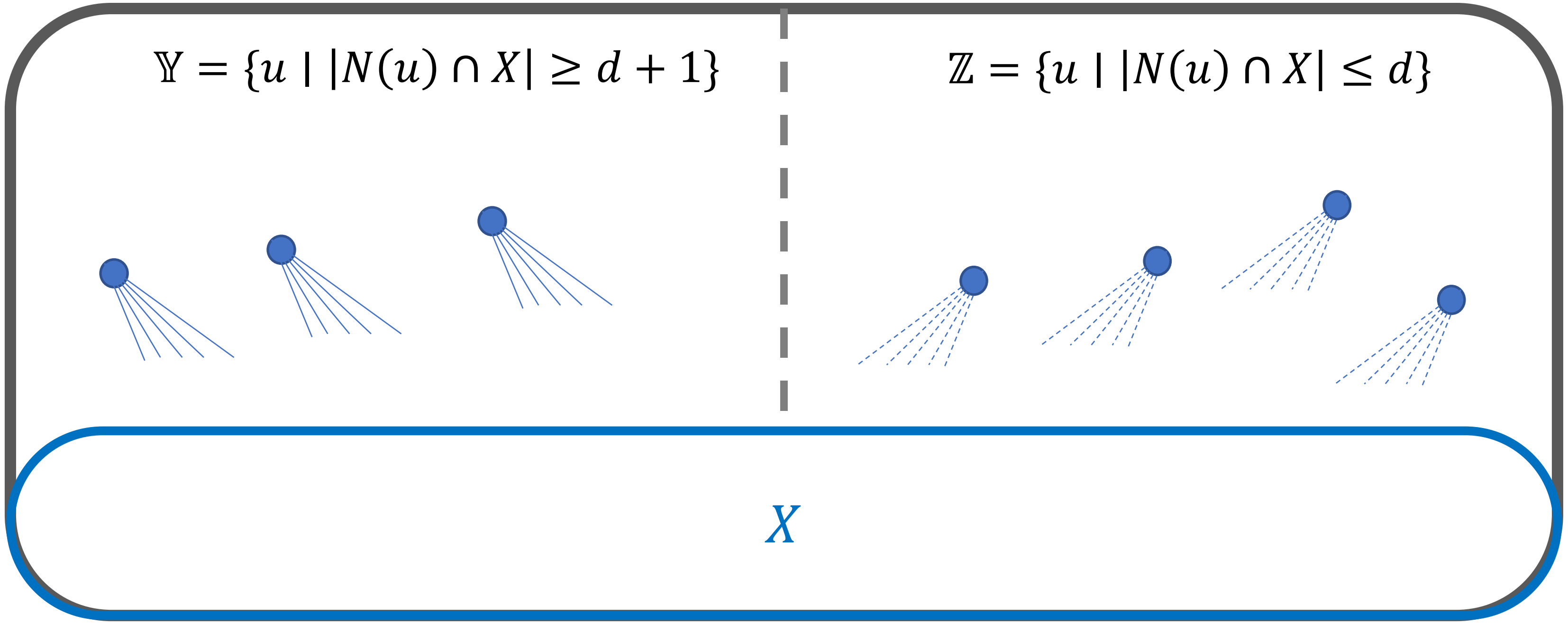
Algorithm:

1. Try every vertex set X of size $\leq 2d + 1$
   a) Try every vertex set Y of size $\leq \binom{2d+1}{d+1} \cdot d$
      i. Try every vertex set Z of size $\leq \binom{2d+1}{d+1} \cdot d$
         A. If $X \cup \bar{Y} \cup Z$ is a solution, then huzzah!

d-degenerate

1. Try X of size $\leq 2d + 1$
   a) Try Y of size $\leq \binom{2d+1}{d+1} \cdot d$
      i. Try Z of size $\leq \binom{2d+1}{d+1} \cdot d$
         A. Check $X \cup \bar{Y} \cup Z$

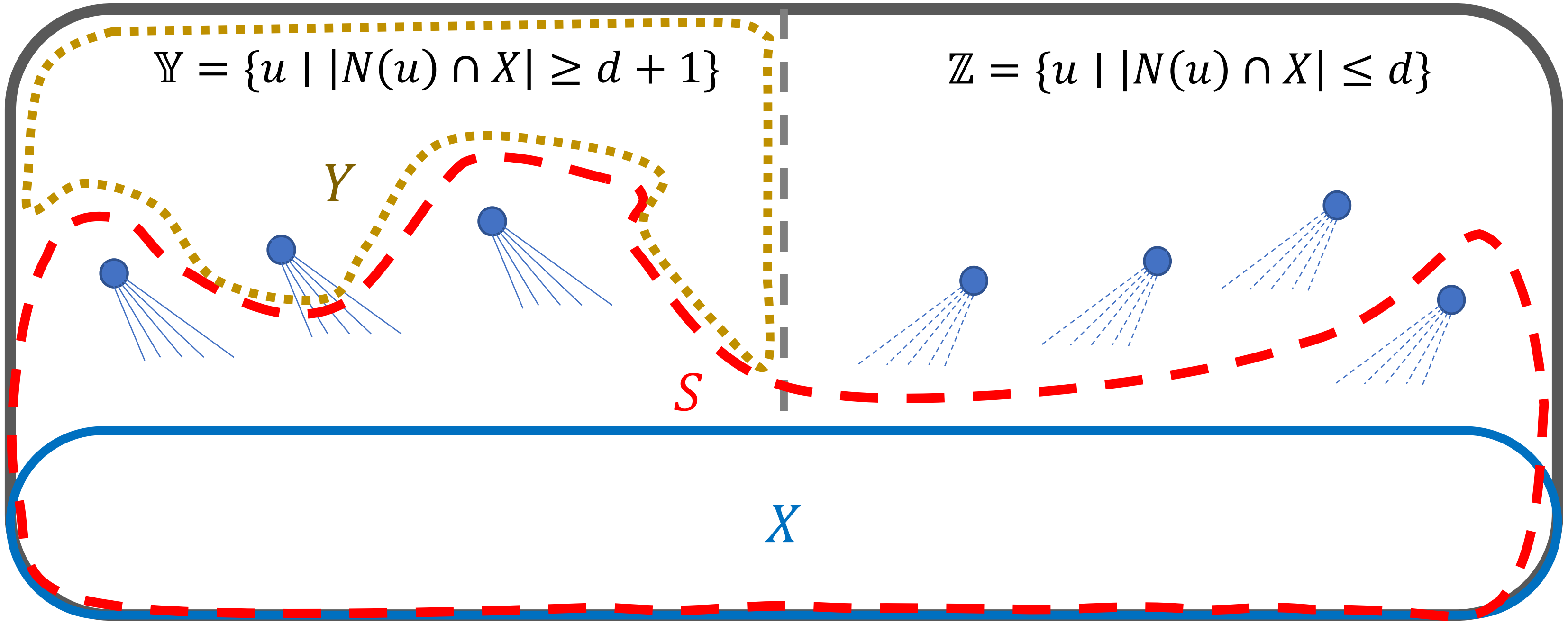
- If solution S has size $\leq 2d$, it will be found with $X = S,\ Y = \emptyset,\ Z = \emptyset$
- Otherwise, guess $X \subseteq S$ of size $2d + 1$



d-degenerate

1. Try X of size $\leq 2d + 1$
   a) Try Y of size $\leq \binom{2d+1}{d+1} \cdot d$
      i. Try Z of size $\leq \binom{2d+1}{d+1} \cdot d$
         A. Check $X \cup \bar{Y} \cup Z$

- A solution S
- $Y = \mathbb{Y} \setminus S$ can't be too big



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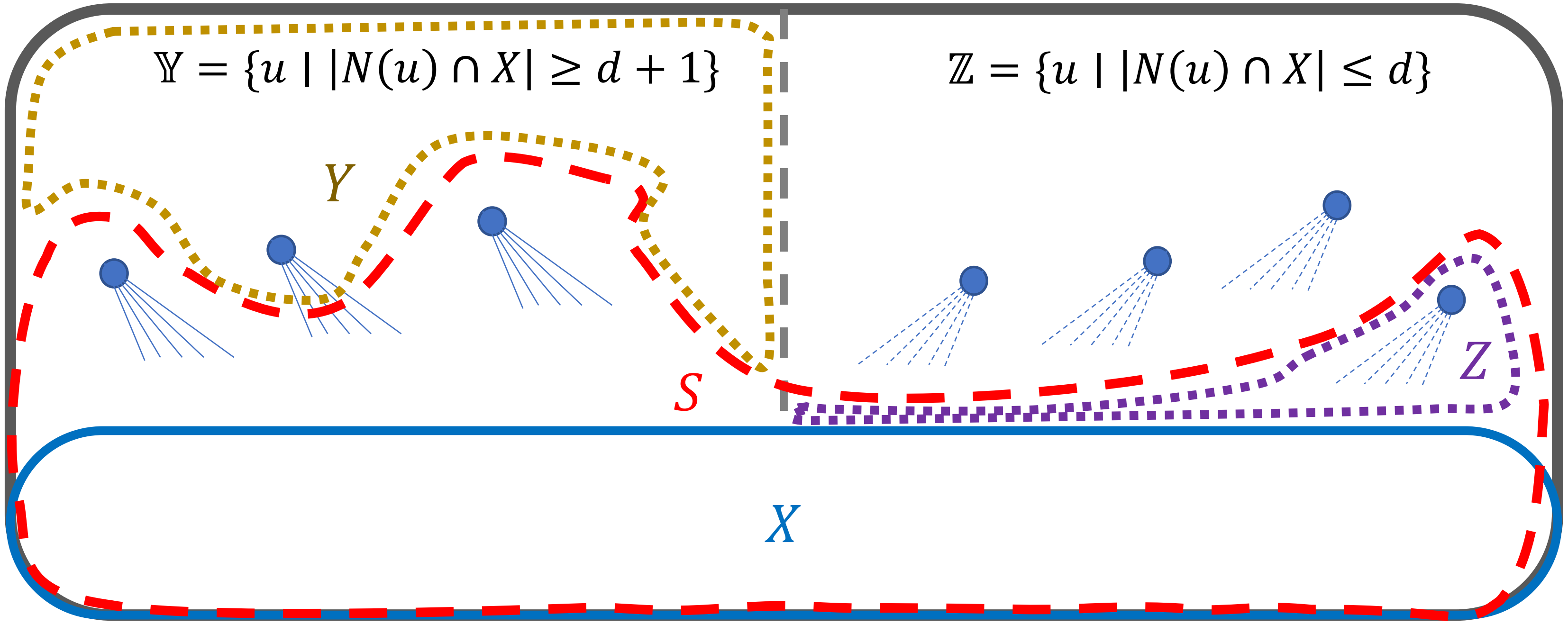
         A. Check $X \cup \bar{Y} \cup Z$

d-degenerate

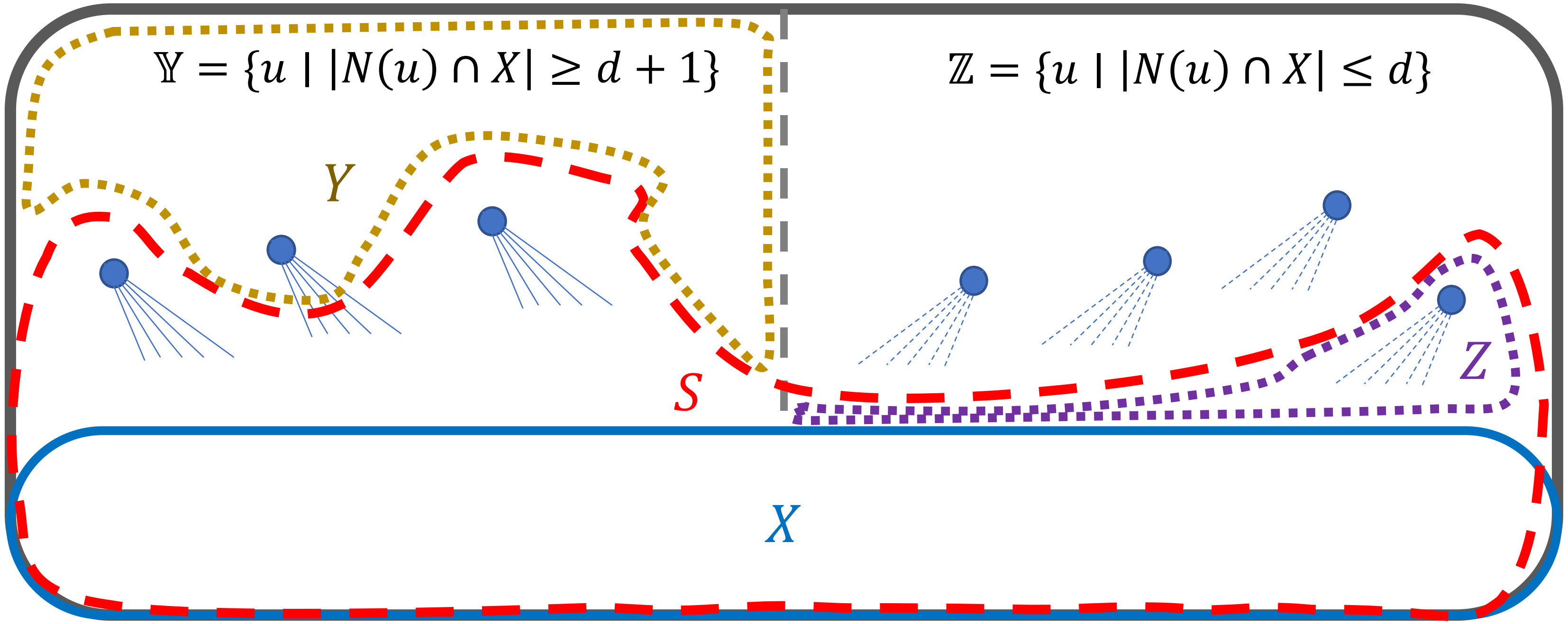
- A solution S
- $Z = \mathbb{Z} \cap S$ can't be too big



d-degenerate

1. Try X of size $\leq 2d + 1$
   a) Try Y of size $\leq \binom{2d+1}{d+1} \cdot d$
      i. Try Z of size $\leq \binom{2d+1}{d+1} \cdot d$
         A. Check $X \cup \bar{Y} \cup Z$

- Runtime $\mathcal{O}(n^{2d+1+2^{2d+1} \cdot 2d} \cdot f(n))$



d-degenerate

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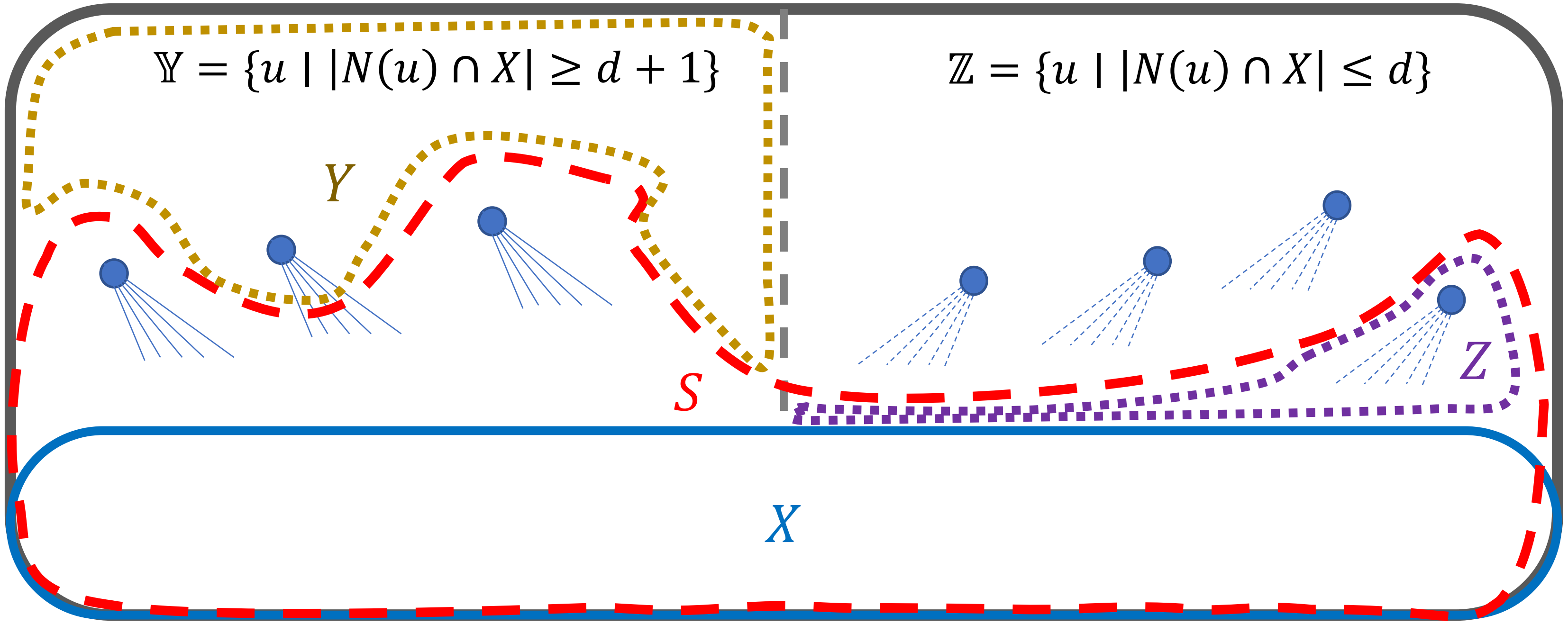
   a) Try Y of size $\leq \binom{2d+1}{d+1} \cdot d$
      i. Try Z of size $\leq \binom{2d+1}{d+1} \cdot d$
         A. Check $X \cup \bar{Y} \cup Z$

- Runtime $n^{\mathcal{O}^*(4^d)} \cdot f(n)$



Regular

Theorem
Subgraph complementation to regular graphs is NP –complete.

Proof:
Reduction from Clique in regular graphs

Regular

Clique in regular graphs

Input:
- A graph G which is r-regular, and
- integer k.

Question:
Does G contain a clique on k vertices?

Partial complement to regular graphs

Input:
- A graph G

Question:
Does G have a partial complement which is regular?

NP –complete [Garey and Johnson 1979]

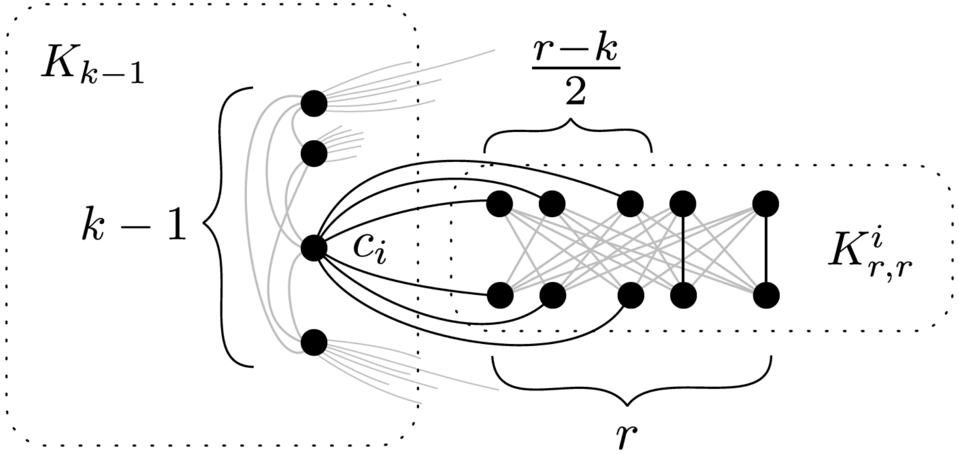
Regular

Reduction algorithm

Input: Instance (G, k) of clique in r-regular graph
Output: Instance G' of partial complement to regular

1. Create the gadget $H_{k,r}$
2. Let the graph G' be the disjoint union of G and $H_{k,r}$

* Some details left out

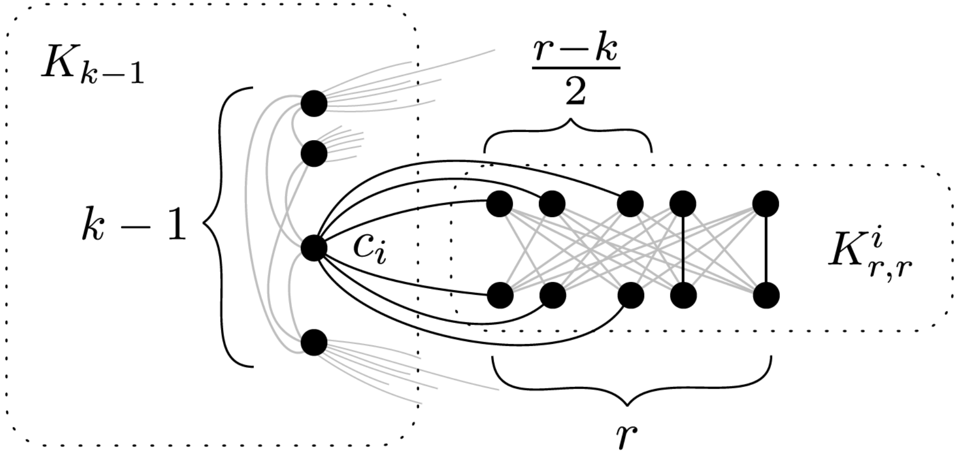


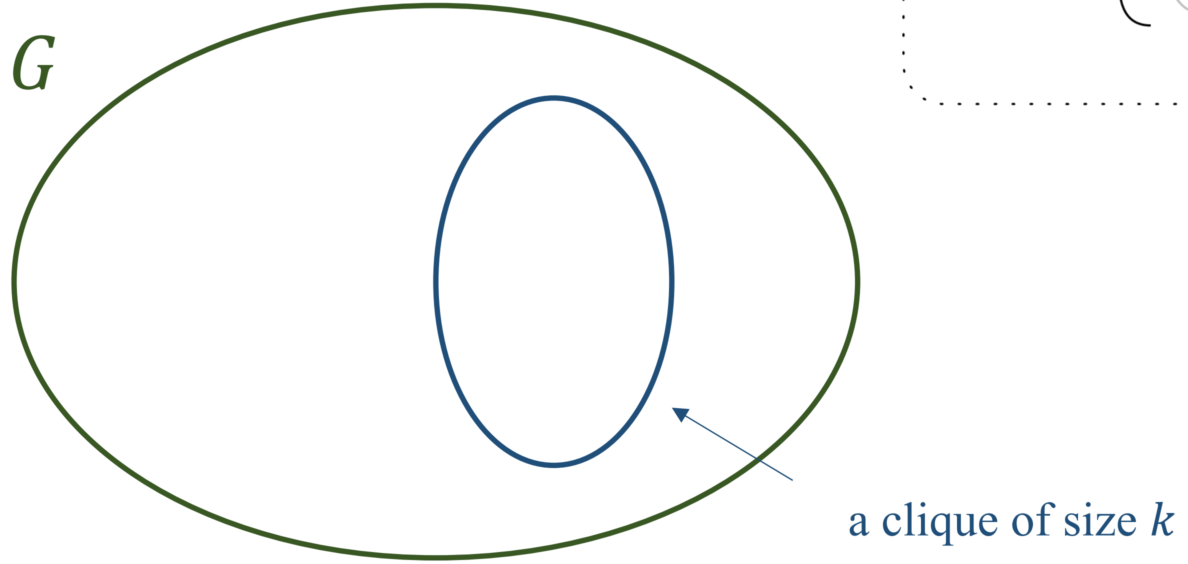
Regular

G contains a clique of size k

$\Downarrow$

G' can be partially complemented to regular graph





Regular

G contains a clique of size k

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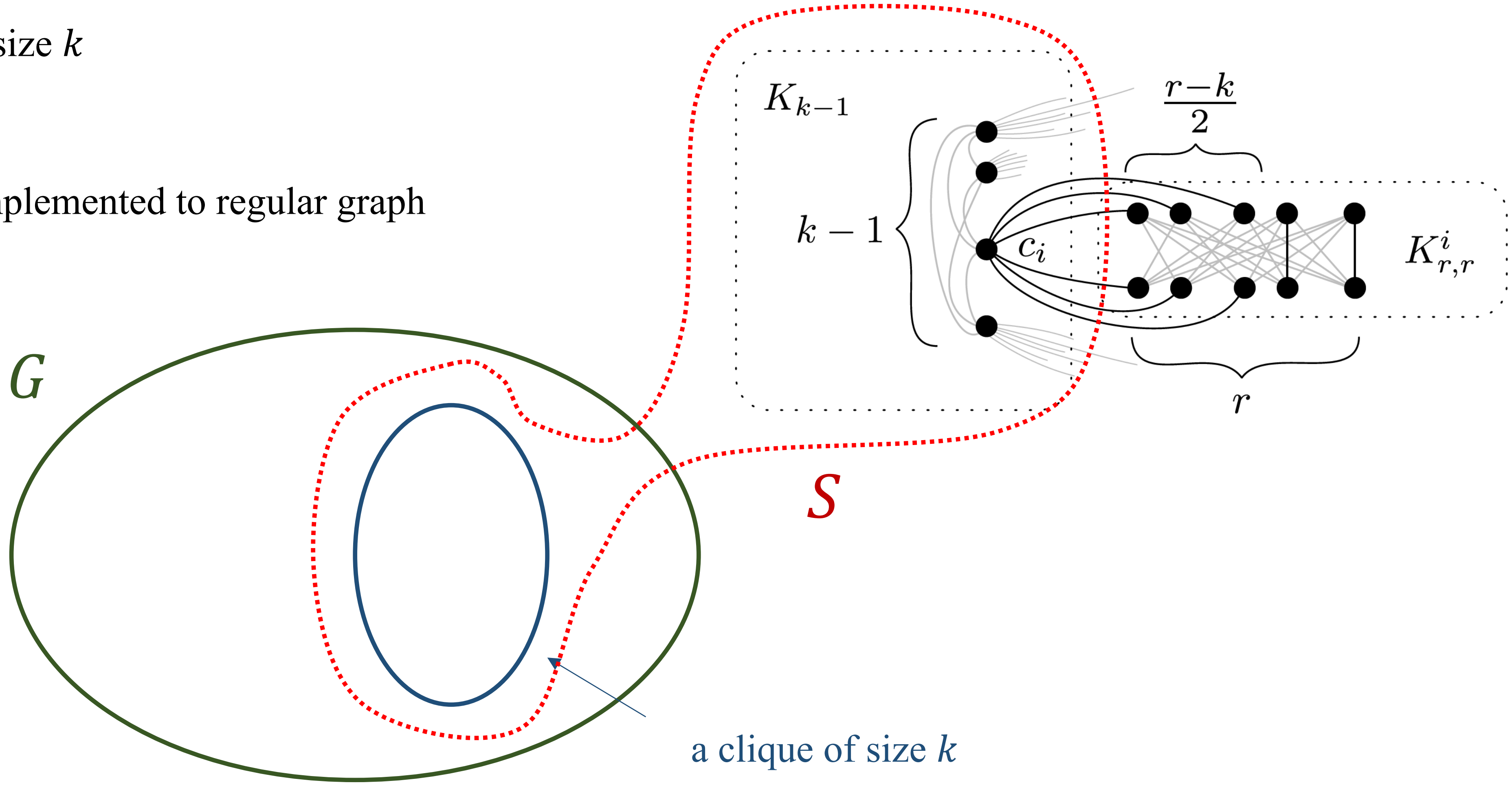
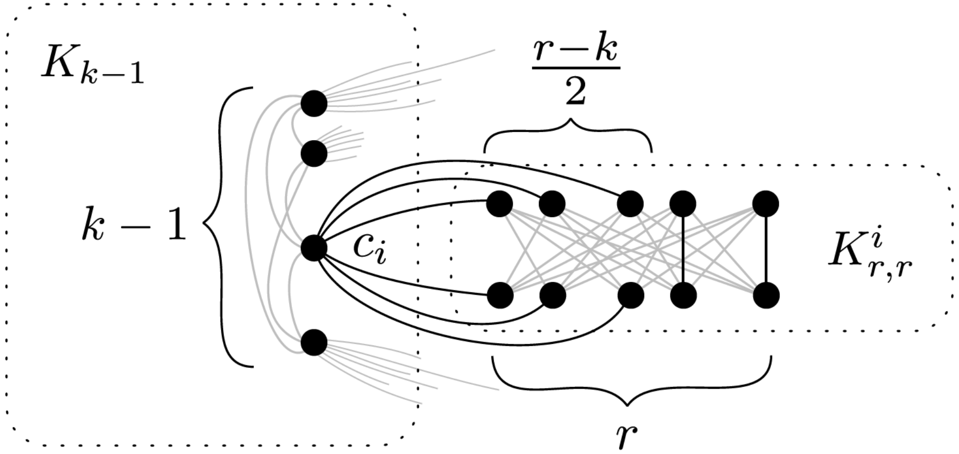


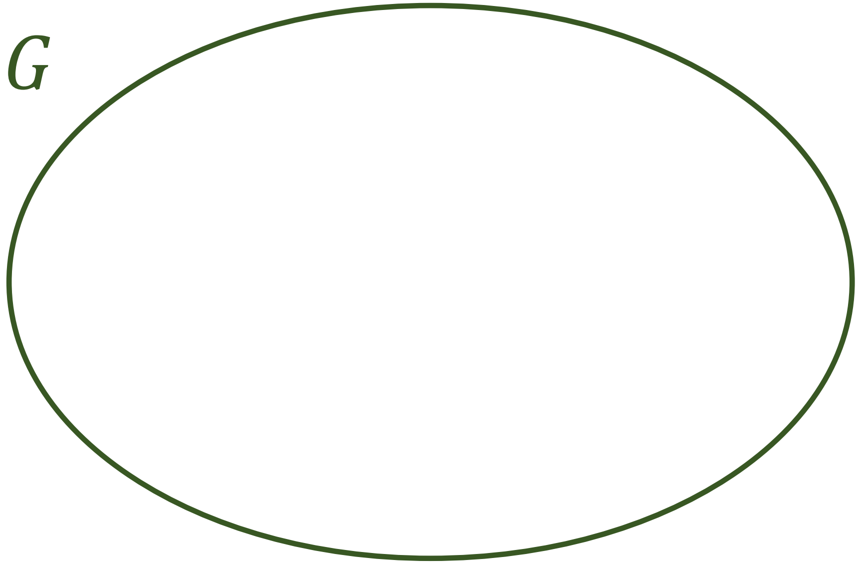
Regular

G' can be partially complemented to regular graph

$$\Downarrow$$

G contains a clique of size k





Regular

G' can be partially complemented to regular graph

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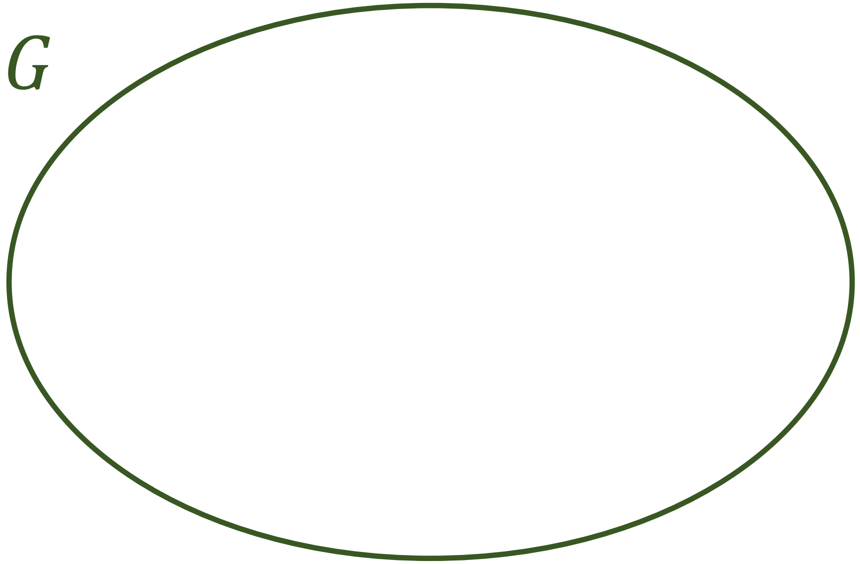
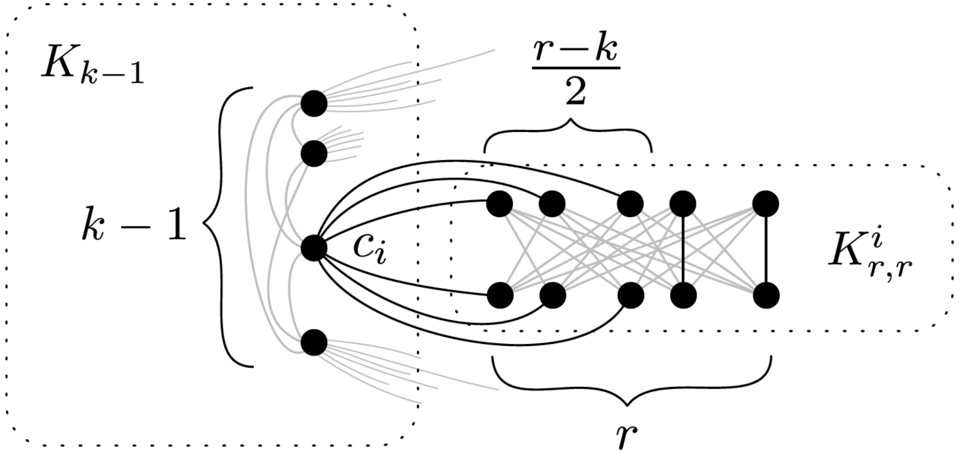




Say we have solution S such that $G \oplus S$ has regularity r'.

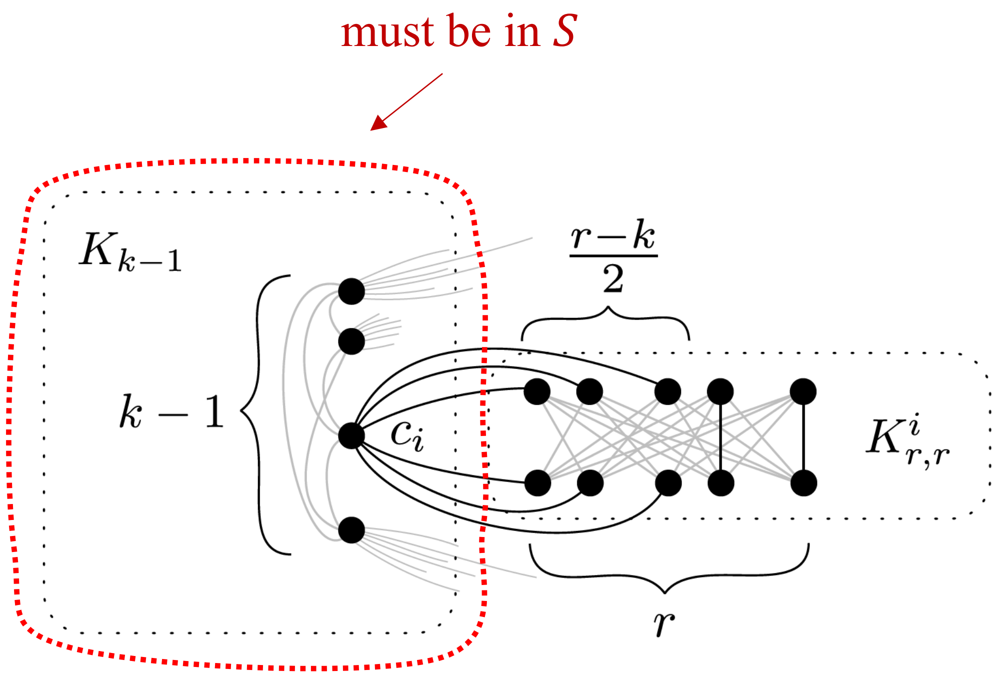
r' = r is only possibility.

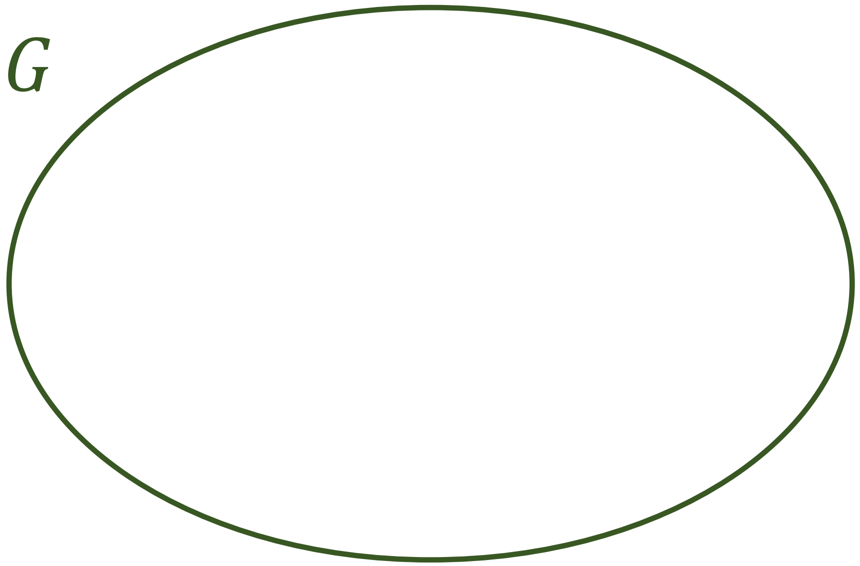
Regular

G' can be partially complemented to regular graph

$\Downarrow$

G contains a clique of size k



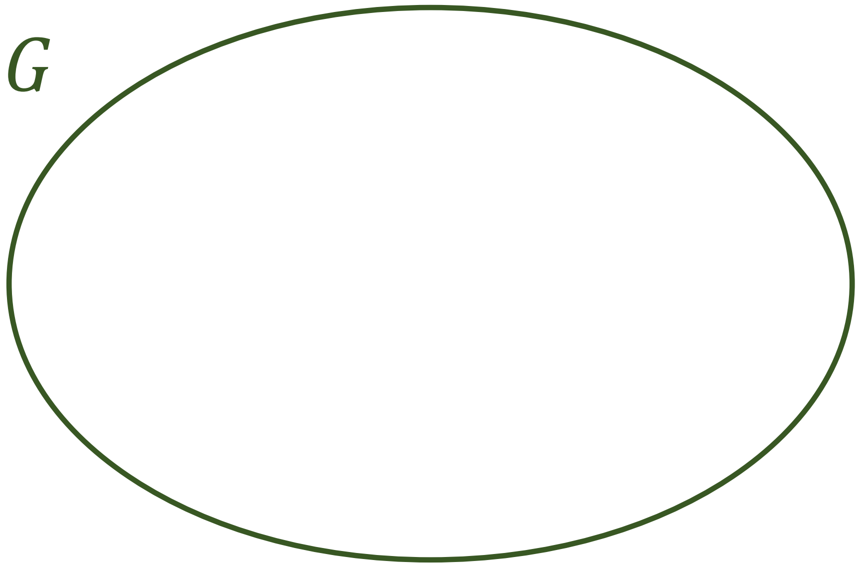


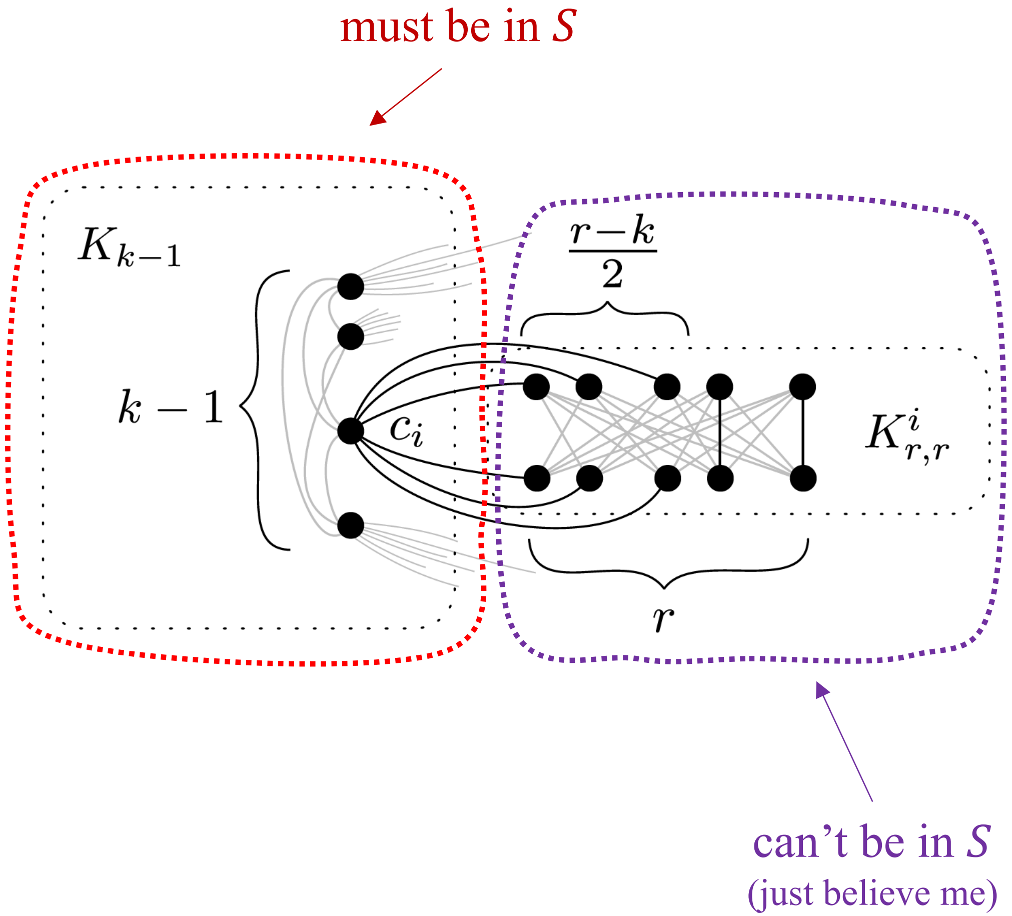
Regular

G' can be partially complemented to regular graph

$\Downarrow$

G contains a clique of size k



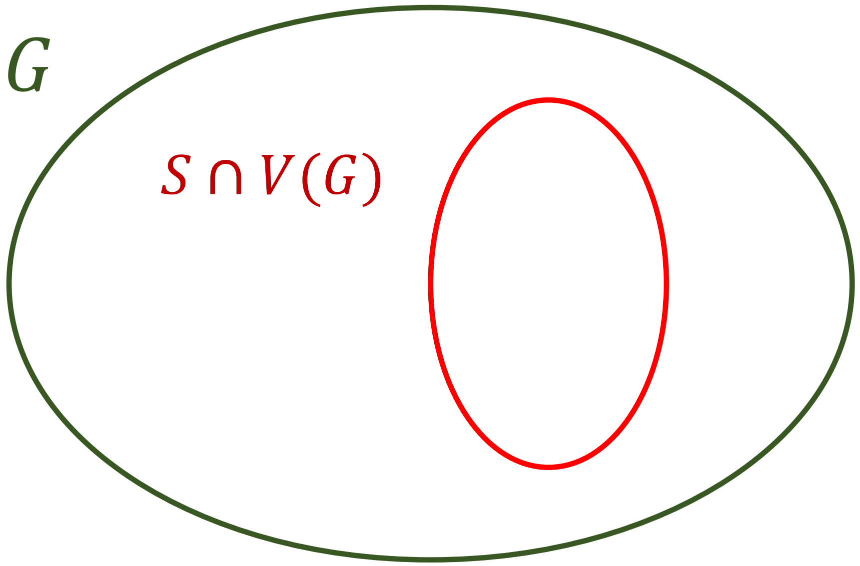


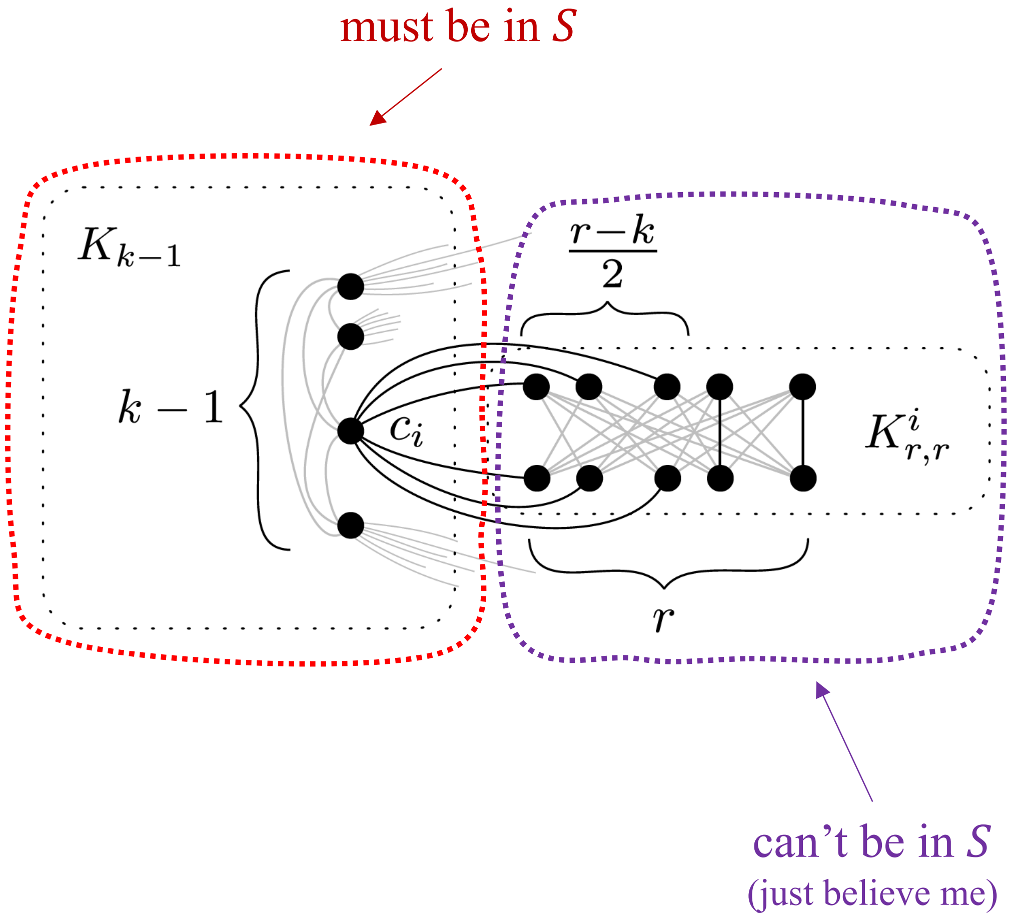
Regular

G' can be partially complemented to regular graph

$\Downarrow$

G contains a clique of size k



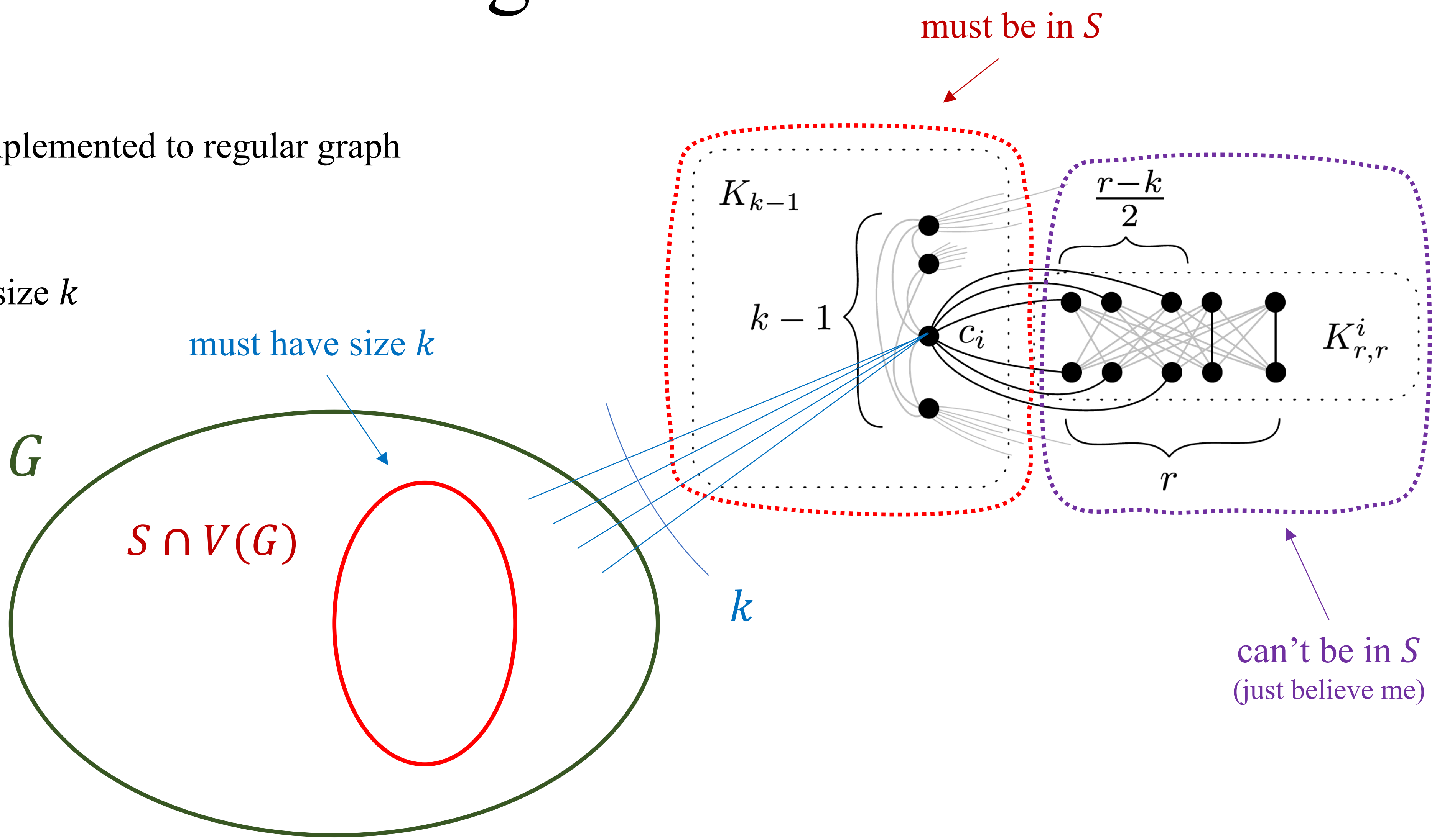


Regular

G' can be partially complemented to regular graph

$\Downarrow$

G contains a clique of size k

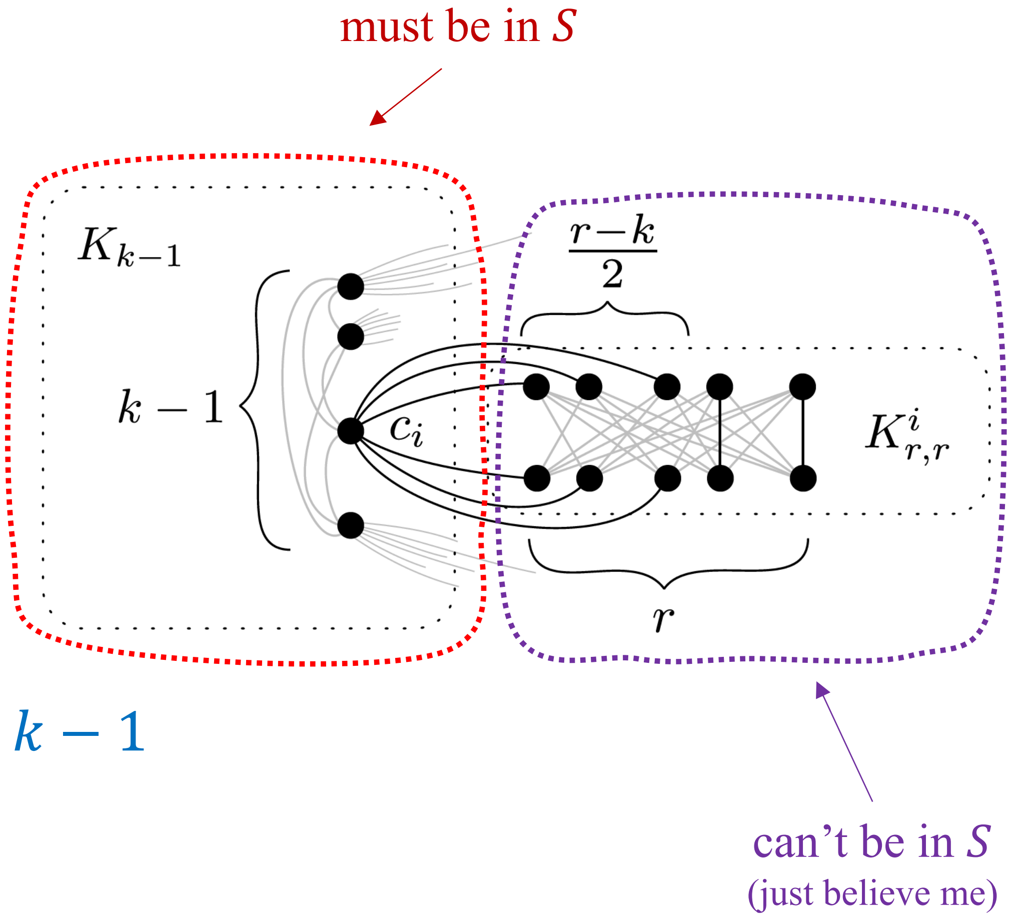


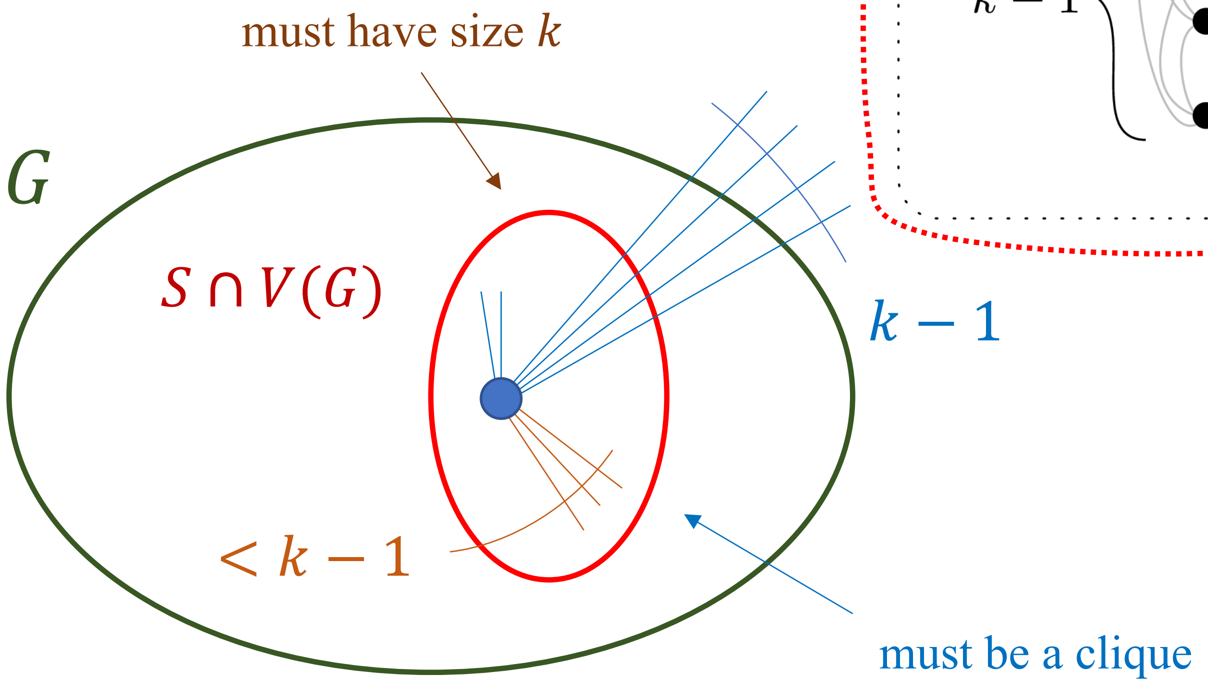
r-regular

G' can be partially complemented to r-regular graph

$\Downarrow$

G contains a clique of size k





Open questions

- What about other graph classes?
  - Chordal
  - Interval
  - P_5 –free
- More than one partial complement?
- More than one edit operation?

Thank you