Computing Independent Transversals for H-free Graphs of Bounded Diameter

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3-Colouring

Definition

A graph G is 3-colourable if and only if there exists a map $c : V(G) \to \{1, 2, 3\}$ such that for any two adjacent vertices, u and v, $c(u) \neq c(v)$.

Alternatively:

A graph G is 3-colourable if and only if there exists an independent set of vertices of G whose removal leaves the remaining graph bipartite.

Independent Odd Cycle Transversal

Definition

A graph G has an independent odd cycle transversal of size at most k if and only if there exists an independent set of at most k vertices whose removal leaves the graph bipartite.

Alternatively:

A graph G has an independent odd cycle transversal of size at most k if and only if it has a 3-colouring such that some colour class has size at most k.

Near-Bipartiteness

Definition

A graph G is near-bipartite if and only if it's vertex set can be partitioned into an independent set and a forest.

Alternatively:

G is near-bipartite if and only if there exists a 3-colouring of G such that some pair of colour classes induces a forest.

Independent Feedback Vertex Set

Definition

G has an independent feedback vertex set of size at most k if and only if there exists an independent set of size at most k whose removal leaves a forest.

Alternatively:

G has an independent feedback vertex set of size at most k if and only if there exists a 3-colouring of G where one colour class has size at most k and the other two induce a forest.

Complexity

- Each of these problems is NP-complete in general.
- We consider the effect of bounding the diameter of the input graph, as well as restricting to classes of H-free graphs.

3-Colouring H-free graphs

Theorem (Holyer, 1981)

3-Colouring is NP*-complete for claw-free graphs.*

Theorem (Emden-Weinert, Hougardy and Kreuter, 1998)

3-Colouring is NP*-complete for graphs of girth at least g for any g.*

Theorem (Bonomo, Chudnovsky, Maceli, Schaudt, Stein and Zhong, 2018)

3-Colouring is polynomial-time solvable for P_7-free graphs.

Independent Odd Cycle Transversal for H-free graphs

Corollary

IOCT is NP*-complete for H-free graphs where H contains a claw or a cycle by reduction from 3-Colouring for H-free graphs.*

Theorem (Bonamy, Dabrowski, Feghali, Johnson and Paulusma, 2018)

Independent Odd Cycle Transversal is polynomial-time solvable for P_5-free graphs

Near-Bipartiteness for H-free Graphs

Theorem (Bonamy, Dabrowski, Feghali, Johnson and Paulusma, 2018)

Near-Bipartiteness is NP*-complete for claw-free graphs*

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Near-Bipartiteness is NP*-complete for graphs of girth at least g for any g.*

Theorem (Bonamy, Dabrowski, Feghali, Johnson and Paulusma, 2018)

Near-Bipartiteness is polynomial-time solvable for P_5-free graphs.

Independent Feedback Vertex Set for H-free Graphs

Corollary

IFVS is NP*-complete for H-free graphs where H contains a claw or a cycle by reduction from near-bipartiteness for H-free graphs*

Theorem (Bonamy, Dabrowski, Feghali, Johnson and Paulusma, 2018)

Independent Feedback Vertex Set is polynomial-time solvable for P_5-free graphs

Diameter

Definition

The distance between two vertices u and v in a graph G is the length of a shortest path between them. The diameter of G is the maximum over all pairs u, v of the distance between u and v.

- What happens when we bound the diameter of G?

Diameter 2

Theorem (Bonamy, Dabrowski, Feghali, Johnson and Paulusma, 2018)

Independent Feedback Vertex set is polynomial-time solvable for graphs of diameter at most 2.

Open Problems

Determine the complexity of 3-Colouring and Independent Odd Cycle Transversal for graphs of diameter 2

Diameter 3

Theorem (Mertzios and Spirakis, 2016)

3-*Colouring is* NP-*complete for graphs of diameter at most* 3*.*

Theorem (Bonamy, Dabrowski, Feghali, Johnson and Paulusma, 2018)

Near-Bipartiteness is NP-*complete for graphs of diameter at most* 3*.*

H-free Graphs of Bounded Diameter

- What if we both restrict the input to classes of H-free graphs and bound the diameter?

Polyads

Definition

A polyad is a tree where exactly one vertex has degree greater than 2. $K_{1,r}^{\ell}$ denotes the star with r leaves and one edge subdivided ℓ times.

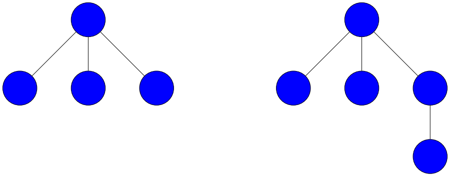


Figure: The claw $K_{1,3}$ and the chair $K_{1,3}^{1}$

$K_{1,r}$-free Graphs

Theorem (Martin, Paulusma, S. , 2019)

3-Colouring is constant-time solvable for $K_{1,r}$-free graphs of diameter at most d for any integers d and r.

- 3-colourable $k_{1,r}$-free graphs have maximum degree at most $R(3, r)$.
- Combining this with bounded diameter gives a constant-sized bound on the number of vertices.
- Hence all four problems are polynomial-time solvable for $k_{1,r}$-free graphs of diameter at most d.

Chair-free graphs

Theorem (Martin, Paulusma, S. , 2019)

3-Colouring is Polynomial time solvable for $K^1_{1,3}$-free graphs of Diameter at most d for any d.

Corollary

We can extend this result to near-bipartiteness, independent feedback vertex set and independent odd cycle transversal.

Proof Outline

- Our aim is to show that each proper 3-colouring of some constant-sized set of vertices Γ leads to exactly one possible 3-colouring of G.
- It remains to check whether each of these colourings is proper, and if so whether it satisfies the conditions for Near-Bipartiteness, Independent Feedback Vertex Set or Independent Odd Cycle Transversal.

Proof

- We begin with the case where G contains an odd cycle C_p with $5 \leq p \leq 2d + 1$.
- Call the vertices of some such cycle, C_p, N_0 and partition the remaining vertices into sets N_i of those at minimum distance i from N_0.

Proof

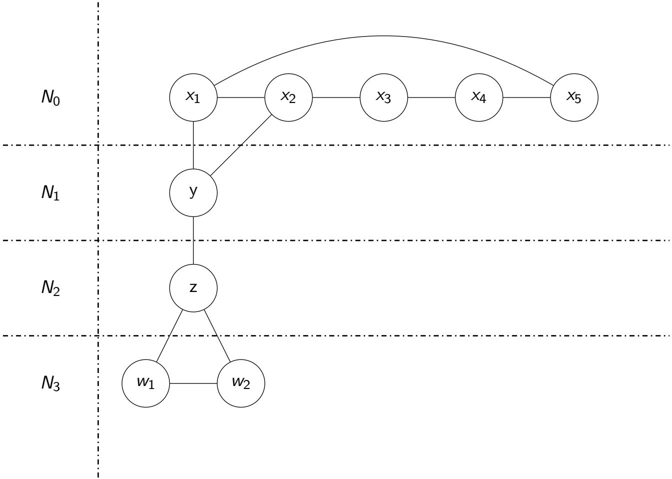
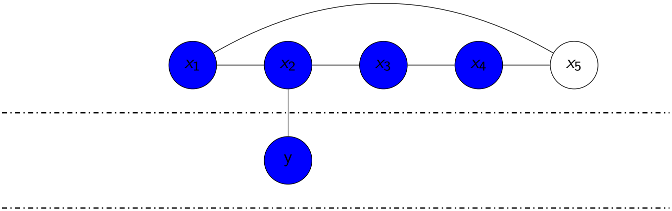


Figure: The case where G contains an odd cycle of length at least 5 and at most $2d+1$.

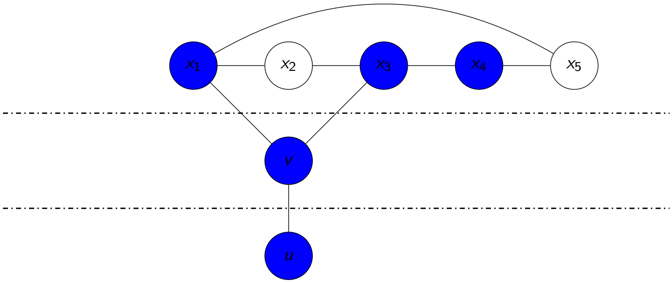
Proof

- To avoid the chair, each vertex of N_1 must have some pair of adjacent parents in N_0. This leaves at most one available colour.



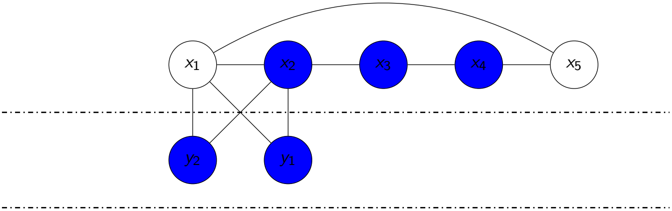
Proof

- If G is 3-colourable, any vertex v in N_1 with an N_2 child has exactly two adjacent parents
- If v has two non-adjacent parents then since G is chair-free v dominates N_0 and G is not 3-colourable



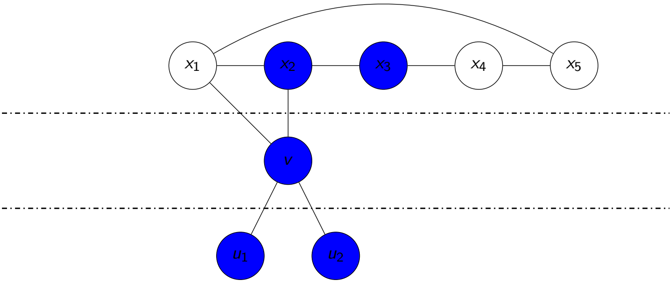
Proof

- Next we show that the number of N_1 vertices with exactly two N_0 parents is at most five.
- If two adjacent vertices share two independent children with no other N_0 neighbours, we get an induced chair.
- Hence, if G is 3-colourable, each pair shares at most one such child.



Proof

- Now we show that the number of vertices in N_i, $i > 1$ is bounded.
- No vertex of N_1 has two independent N_2 children, hence there are at most ten vertices in N_2.



proof

- Similarly no vertex in N_i for $i \geq 2$ has two independent children in N_{i+1}.
- Hence there are at most $5 \times 2^{i-1}$ vertices in N_i for $i \geq 2$.

Proof

- Let Γ be the set $N_0 \cup \{N_i : i \geq 2\}$
- Γ has size at most $5 + \sum_{i=2}^{d} 5 \times 2^{i-1}$
- Any 3-colouring of Γ corresponds to at most one possible 3-colouring of G, we simply check whether any such colouring is proper.

Proof

- For Near-Bipartiteness we also check whether any two colour classes in some proper colouring form a forest.
- For IFVS we check for such a colouring where the third colour class has size at most k.
- For IOCT, we check whether any proper 3-colouring has a colour class of size at most k.

Triangle-free Graphs

Theorem (Mertzios and Spirakis, 2016)

3-*Colouring is* NP-*complete for triangle-free graphs of diameter at most* 3.

- The construction of Mertzios and Spirakis is 3-colourable if and only if it is near-bipartite.
- Hence all four problems are NP-complete for triangle-free graphs of diameter at most 3.

Diameter and Girth

Definition

The girth of a graph G is the length of its shortest cycle. In other words G has girth g if and only if G is $(C_3, C_4, ..., C_{g-1})$-free.

Diameter and Girth

Theorem (Damerell, 1973, Hoffman and Singleton, 1960, Singleton, 1968)

For every $d \geq 1$, every graph of diameter d and girth $2d+1$ is p-regular for some integer p. Moreover, if $d = 2$, then there are only four such graphs (with $p = 2, 3, 7, 57$, respectively) and if $d \geq 3$, then such graphs are cycles (of length $2d+1$).

Corollary

3-Colouring, IOCT, Near-Bipartiteness and IVFS are constant-time solvable for graphs of diameter d and girth $2d+1$ for any integers k and d.

Diameter and Girth

Theorem (Martin, Paulusma and S. 2019)

3-Colouring is NP*-complete for graphs of diameter at most* $4p$ *and girth at least* $4p + 2$ *for any integer* p*.*

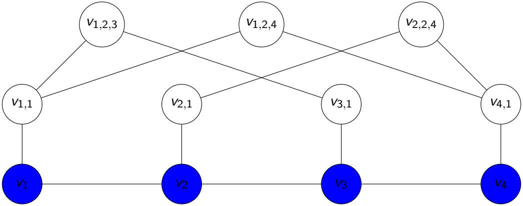


Figure: The reduction for $p = 1$

Corollary

This implies NP*-completeness of IOCT and can be extended to Near-Bipartiteness and IFVS.*

Summary of Results for Diameter and Girth

diameter \ girth	≥ 3	≥ 4	≥ 5	≥ 6	≥ 7	≥ 8	≥ 9	≥ 10	≥ 11
≤ 1	P	P	P	P	P	P	P	P	P
≤ 2	?	?	P	P	P	P	P	P	P
≤ 3	NP-c	NP-c	?	?	P	P	P	P	P
≤ 4	NP-c	NP-c	NP-c	NP-c	?	?	P	P	P
≤ 5	NP-c	NP-c	NP-c	NP-c	?	?	?	?	P

Figure: The complexity of 3-Colouring for graphs of diameter at most d and girth at least g.

Open Problems

- 3-Colouring / Independent Odd Cycle Transversal for (Triangle-free) graphs of diameter at most two.
- What are the complexities of all four problems for graphs of diameter d and girth g, $g \leq 2d$.
- What are the complexities of Near-Bipartiteness, Independent Feedback Vertex Set and Independent Odd Cycle Transversal for P_6-free graphs.
- What are the complexities of all four problems for $K_{1,3}^2$-free graphs of diameter 3. or $K_{1,4}^1$-free graphs of diameter 3.

Thank you!