Preprocessing Vertex-Deletion Problems:

Characterizing Graph Properties by Low-Rank Adjacencies

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Vertex-Deletion Problem

Π-free Deletion
Input: A graph G and an integer k.
Question: Does there exist a set $S \subseteq V(G)$ of size at most k such that $G - S$ does not contain any graph from Π as induced subgraph?

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Question

Can we efficiently reduce the size of the input graph without changing the answer?

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Analyze problems in terms of input size *and* in terms of an additional parameter.

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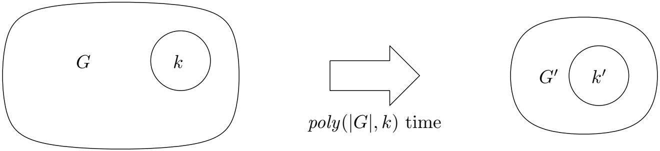
Kernelization

Efficiently reduce an instance (G, k) to an equivalent instance (G', k') of size bounded by some $f(k)$.



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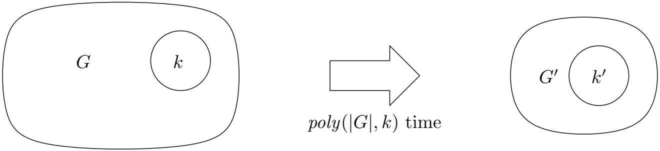

If $f(k)$ is polynomial function, (G', k') is polynomial kernel.

Running Example: Perfect Deletion

Perfect graph

- Chromatic number of every induced subgraph equals its largest clique size.

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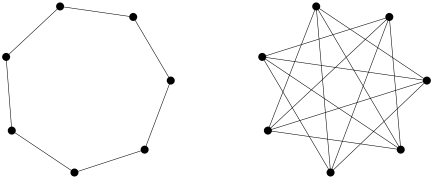
- Graph without induced cycle (hole) of odd length at least 5 or its edge complement.



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- Polynomial kernel for INTERVAL DELETION (K) - Agrawal et al. [2019].

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Perfect Deletion (vc) **Parameter:** $|X|$
Input: A graph G, a vertex cover X of G, and an integer k.
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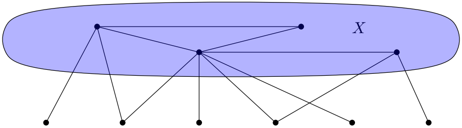

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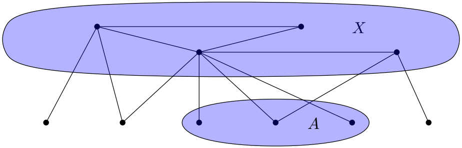

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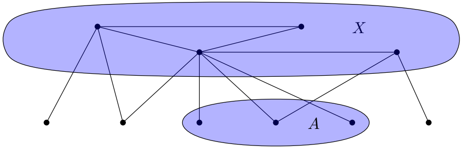

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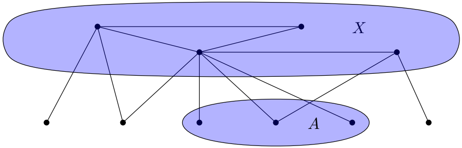

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- Challenge: Pick A so that other direction holds.

Incidence vectors

Intuition: set A should represent independent set $V(G) \setminus X$.

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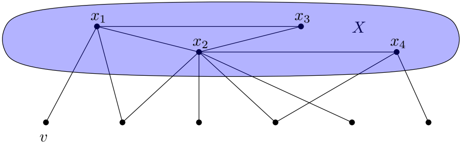

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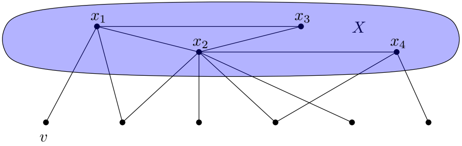

$$\mathrm{inc}_{(G,X)}(v) = \begin{pmatrix} x_1: & 1 \\ x_2: & 0 \\ x_3: & 0 \\ x_4: & 0 \end{pmatrix}$$

Basic incidence vector

For $x_i \in X$,

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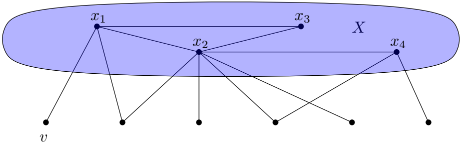

$$\mathrm{inc}^2_{(G,X)}(v) = \begin{pmatrix} (\{x_1\}, \emptyset) & : & 1 \\ (\{x_1, x_2\}, \emptyset) & : & 0 \\ (\{x_1\}, \{x_2\}) & : & 1 \\ \vdots & & \vdots \end{pmatrix}$$

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More general (rank-c incidence vector)

For disjoint $P, Q \subseteq X$ s.t. $|P| + |Q| \leq c$,

$\mathrm{inc}^c_{(G,X)}(v)[(P,Q)] = 1$ iff $P \subseteq N(v)$ and $Q \cap N(v) = \emptyset$.

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Marking set A.

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Claim

Resulting graph has $\mathcal{O}(|X| + (k + 2|X| + 1) \cdot |X|^4) = \mathcal{O}(|X|^5)$ vertices.

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Lemma

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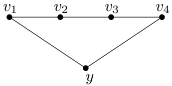

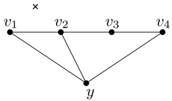

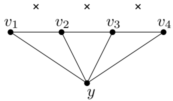

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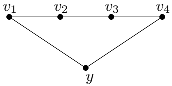

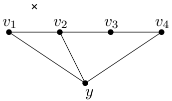

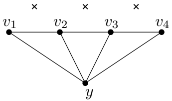

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- y sees v_2 (and v_3) $\rightarrow$ 1 edge (3 edges) seen.

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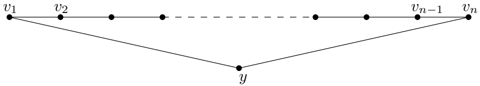

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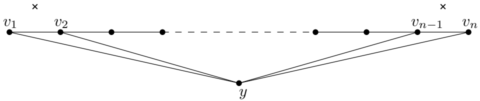

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- y sees first and last edge $\rightarrow$ IH on $P' = \{v_2, ..., v_{n-1}\}$.

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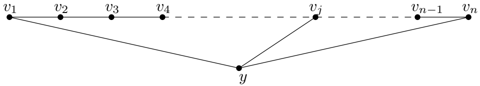

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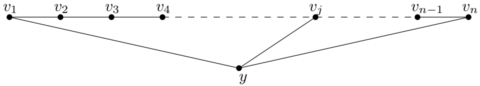

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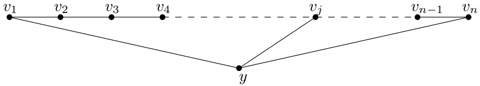

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- If j even, then $j \neq 2 \rightarrow$ IH on $P' = \{v_1, ..., v_j\}$.

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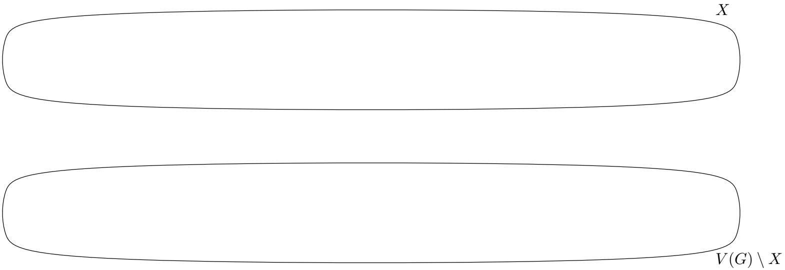
$G[X \cup A] - S$ perfect $\Rightarrow G - S$ perfect



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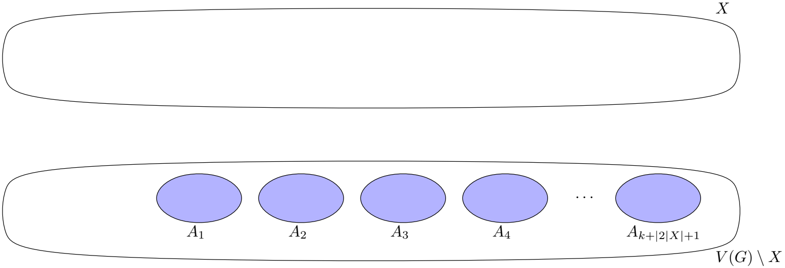

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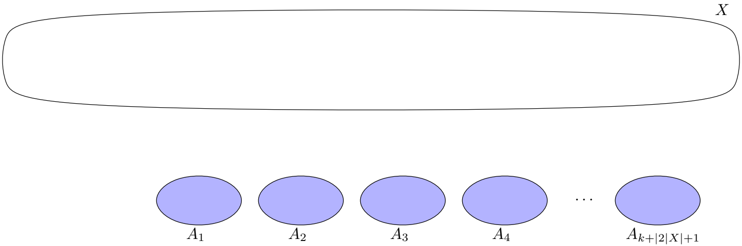

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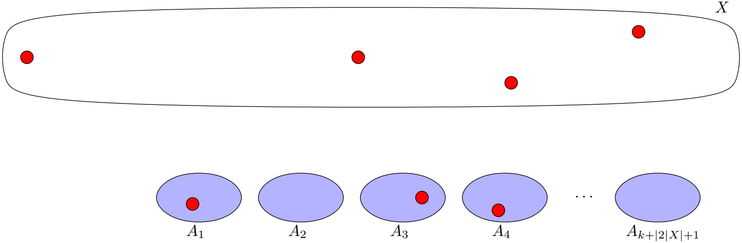

- Compute disjoint bases A_i for $i \in [k + 2|X| + 1]$, $A = \bigcup_i A_i$.
- Consider kernel graph $G[X \cup A]$.
- Consider solution S (red) s.t. $G[X \cup A] - S$ perfect.

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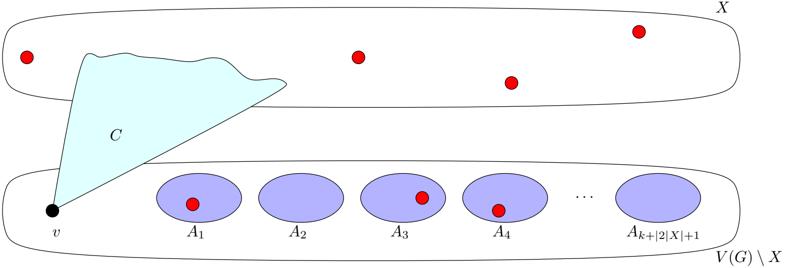
$G[X \cup A] - S$ perfect $\Rightarrow$ $G - S$ perfect



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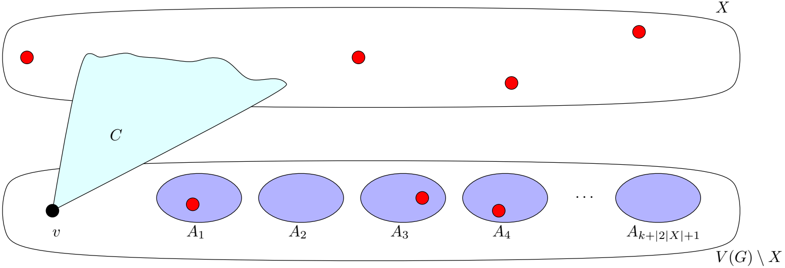

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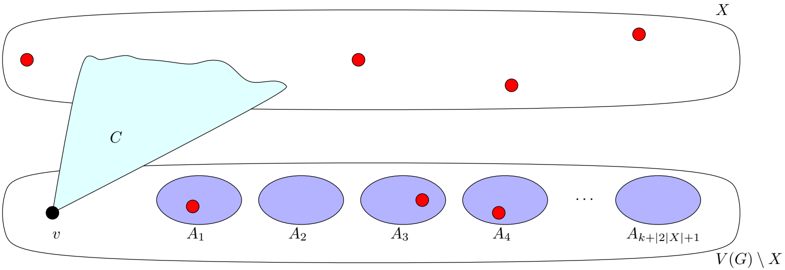

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- Let C be an odd hole s.t. $|V(C) \setminus (X \cup A)|$ minimum.
- $|C| + |S| \leq 2|X| + k$, there exists A_i outside $S \cup C$ (e.g. A_2).

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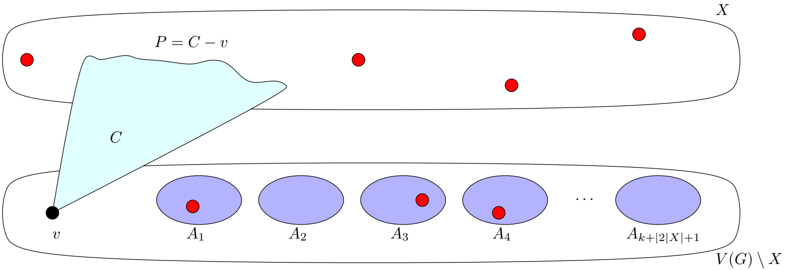

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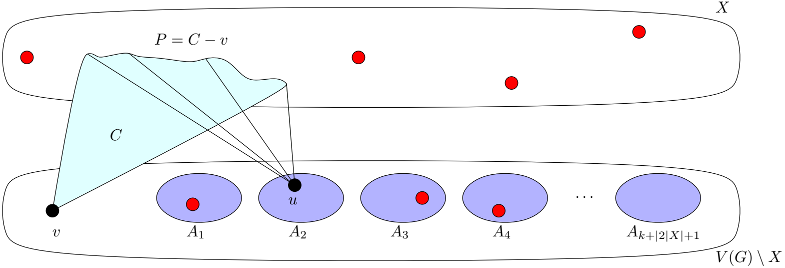

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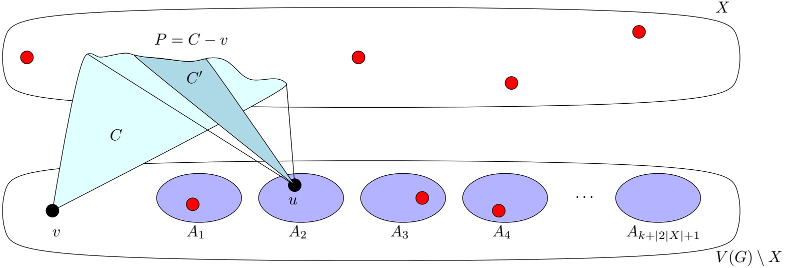

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Running Example: PERFECT DELETION (VC)

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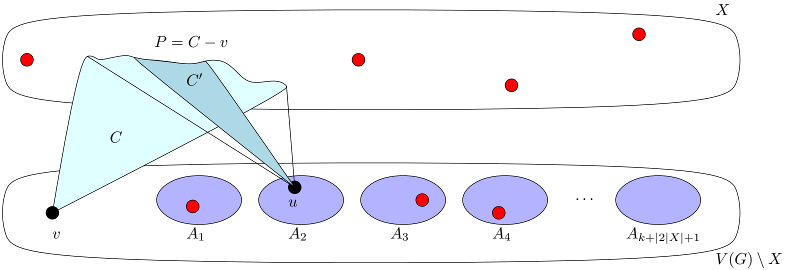

- As v not marked, $\mathrm{inc}^4_{(G,X)}(v) = \sum_{y \in A_i} \mathrm{inc}^4_{(G,X)}(y)$ over $\mathbb{F}_2$.
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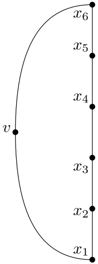
- By lemma, there exists odd hole C' that contradicts minimality of C.
- Hence $G - S$ does not contain odd hole.

Running Example: PERFECT DELETION (VC)

Concrete example



- $X = \{x_1, ..., x_6\}$ and v.

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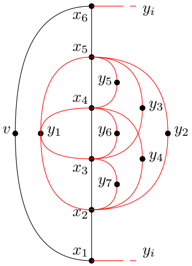

- $X = \{x_1, ..., x_6\}$ and v.
- $Y = \{y_1, ..., y_7\}$.

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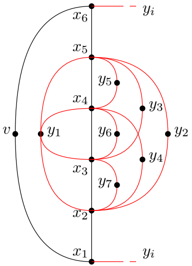

	y_1	y_2	y_3	y_4	y_5	y_6	y_7	v
$(\{x_5\}, \emptyset)$	1	1	1	0	1	0	0	0
$(\{x_1, x_2\}, \emptyset)$	1	1	0	1	0	0	1	0
$(\{x_2, x_3\}, \emptyset)$	1	0	0	0	0	0	1	0
$(\{x_3, x_4\}, \emptyset)$	1	0	0	0	0	1	0	0
$(\{x_1, x_6\}, \{x_3, x_4\})$	0	1	0	0	0	0	0	1
$\vdots$				$\vdots$				$\vdots$

- $X = \{x_1, ..., x_6\}$ and v.
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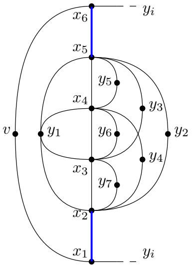

Vertex	$\{p, v_1\}$	$\{v_1, v_2\}$	$\{v_2, v_3\}$	$\{v_3, v_4\}$	$\{v_4, q\}$	$\dots$
y_1	1	1	1	1	1	
y_2	1	0	0	0	1	
y_3	0	0	0	0	1	
y_4	1	0	0	0	0	$\dots$
y_5	0	0	0	1	1	
y_6	0	0	1	0	0	
y_7	1	1	0	0	0	
v	0	0	0	0	0	$\dots$

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- $\mathrm{inc}^4_{(G,X)}(v) = \sum_{y \in Y} \mathrm{inc}^4_{(G,X)}(y)$ over $\mathbb{F}_2$.
- y_2 sees 2 edges of X.

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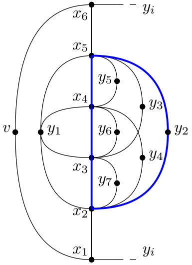

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- $Y = \{y_1, ..., y_7\}$.
- $\mathsf{inc}^4_{(G,X)}(v) = \sum_{y \in Y} \mathsf{inc}^4_{(G,X)}(y)$ over $\mathbb{F}_2$.
- y_2 sees 2 edges of X.
- $G[\{x_2, x_3, x_4, x_5, y_2\}]$ induces odd hole.

Running Example: Perfect Deletion (vc)

Odd anti-holes can be dealt with in a similar fashion.

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Theorem

Perfect Deletion (vc) admits a kernel with $\mathcal{O}(|X|^5)$ vertices.

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Theorem

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For which Π does Π-free Deletion (vc) admit polynomial kernel?

We can generalize this incidence vector approach.

Meta-theorem

Definition (rank-c adjacencies)

Let $c \in \mathbb{N}$. Graph property Π is characterized by rank-c adjacencies if the following holds:

For each graph H, for each vertex cover X of H, for each set $D \subseteq V(H) \setminus X$, for each $v \in V(H) \setminus (D \cup X)$, if

- $H - D \in \Pi$, and
- $\mathrm{inc}^c_{(H,X)}(v) = \sum_{u \in D} \mathrm{inc}^c_{(H,X)}(u)$ when evaluated over $\mathbb{F}_2$,

then there exists $D' \subseteq D$ such that $H - v - (D \setminus D') \in \Pi$.

Meta-theorem

Theorem [Fomin et al. 2014]

If Π is a graph property such that:

(i) Π is characterized by c adjacencies,

(ii) every graph in Π contains at least one edge, and

(iii) there is a polynomial $p\colon \mathbb{N} \to \mathbb{N}$ such that all graphs G that are vertex-minimal with respect to Π satisfy $|V(G)| \leq p(\textsc{vc}(G))$,

then Π-FREE DELETION parameterized by the vertex cover size x admits a polynomial kernel with $\mathcal{O}((x + p(x))x^c)$ vertices.

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Fomin et al. [2014]

$\Pi :=$ all graphs that...	c?	Π-free deletion kernel
contain C_n for some $n \geq \ell$	$\ell - 1$	$\mathcal{O}(\|X\|^{\ell})$ vrtcs
contain an odd cycle	2	$\mathcal{O}(\|X\|^{3})$ vrtcs
...	...	...

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Our results

$\Pi :=$ all graphs that...	rank-c?	Π-free deletion kernel
are not perfect	4	$\mathcal{O}(\lvert X\rvert^{5})$ vrtcs
contain even holes	3	$\mathcal{O}(\lvert X\rvert^{4})$ vrtcs
contain asteroidal triples	8	$\mathcal{O}(\lvert X\rvert^{9})$ vrtcs
are not interval	8	$\mathcal{O}(\lvert X\rvert^{9})$ vrtcs
contain a wheel	4	$\mathcal{O}(\lvert X\rvert^{5})$ vrtcs

Meta-theorem

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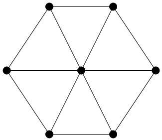
A wheel W_n for some $n \geq 3$ consists of an induced cycle C_n with an apex vertex.



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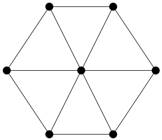

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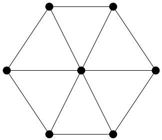

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- Is the meta-theorem tight now?
- Can the meta-theorem be used for Permutation Deletion or Comparability Deletion?