

Preprocessing Vertex-Deletion Problems: Characterizing Graph Properties by Low-Rank Adjacencies

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Workshop on Graph Modification, January 2020

Vertex-Deletion Problem

-free Deletion

Input: A graph G and an integer k .

Question: Does there exist a set $S \subseteq V(G)$ of size at most k such that $G - S$ does not contain any graph from \mathcal{H} as induced subgraph?

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F -free Deletion

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Often stated as F -Deletion.

Vertex-Deletion Problem

F -free Deletion

Input: A graph G and an integer k .

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Example

k - K_2 -free Deletion Vertex Cover

edgeless graphs g -Deletion

Vertex-Deletion Problem

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$F = K_2$ -free Deletion = Vertex Cover

$F =$ edgeless graphs g -Deletion

NP-hard - Lewis and Yannakakis [1980].

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k - K_2 -free Deletion Vertex Cover
 f -edgeless graphs g -Deletion

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Question

Can we efficiently reduce the size of the input graph without changing the answer?

Parameterized complexity

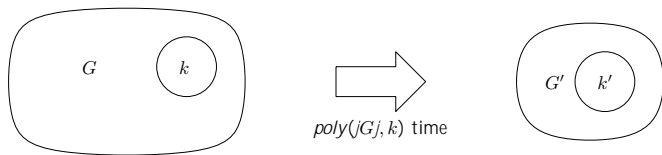
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Kernelization

Efficiently reduce an instance $(G; k)$ to an equivalent instance $(G'; k')$ of size bounded by some $f(k)$.

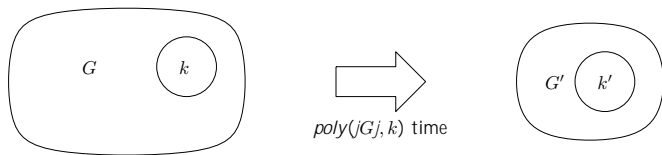


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If $f(k)$ is polynomial function, $(G'; k')$ is polynomial kernel.

Running Example: Perfect Deletion

Perfect graph

- | Chromatic number of every induced subgraph equals its largest clique size.

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Perfect graph

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- | Graph without induced cycle (hole) of odd length at least 5 or its edge complement.

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- | Polynomial kernel for Interval Deletion (k) - Agrawal et al. [2019].

Running Example Perfect Deletion (vc)

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Parameter: $|X|$

Input: A graph G , a vertex cover X of G , and an integer k .

Question: Does there exist a set $S \subseteq V(G)$ of size at most k such that $G - S$ does not contain an odd (anti-)hole as induced subgraph?

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Reduce instance $(G; X; k)$ to equivalent instance $(G[X \cup A]; X; k)$,
s.t. $|A| \leq p(|X|)$.

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- | $(G - S \text{ perfect}) \iff (G[X \cup A] - S \text{ perfect})$.
- | Challenge: Pick A so that other direction holds.

Incidence vectors

Intuition: set A should represent independent set $\forall (G) \cap X$.

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Basic incidence vector

For $x_i \in X$,

$$\text{inc}_{(G;X)}(v)[x_i] = 1 \text{ if } x_i \in N(v).$$

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Basic incidence vector

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More general (rank-c) incidence vector

For disjoint $P, Q \subseteq X$ s.t. $|P| + |Q| = c$,

$$\text{inc}_{(G;X)}^c(v)[(P; Q)] = 1 \text{ if } P \cap N(v) = \emptyset \text{ and } Q \subseteq N(v).$$

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Marking set A.

For any induced cycle C in G with vertex cover X : $|C| \leq 2|X|$.

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Marking Scheme

- 1. Compute multiset of vectors $\text{inc}_{(G;X)}^4(u)$ for $u \in V(G) \setminus X$.

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Marking Scheme

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- | Repeat $k + 2|X| + 1$ times:

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 - | Mark a unique vertex corresponding to each vector in basis.

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- | Remove all unmarked vertices.

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Claim

Resulting graph has $O(|X| + (k + 2|X| + 1) |X|^4) = O(|X|^5)$ vertices.

Running Example Perfect Deletion (vc)

Lemma

Let $P = (v_1, \dots, v_n)$ be a path on vertices where $n - 4$ is even, let y be a vertex not on P such that it is adjacent to both endpoints of P . If y sees an even number of edges of P then the graph contains an odd hole.

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Lemma

Let $P = (v_1, \dots, v_n)$ be a path on vertices where $n \geq 4$ is even, let y be a vertex not on P such that it is adjacent to both endpoints of P . If y sees an even number of edges of P then the graph contains an odd hole.

Proof by induction on: base case

- | y sees no other vertex odd hole (C_5).

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Proof by induction on n : base case

- | y sees no other vertex odd hole (C_5).
- | y sees v_2 (and v_3) ! 1 edge (3 edges) seen.

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Let $P = (v_1, \dots, v_n)$ be a path on vertices where $n \geq 4$ is even, let y be a vertex not on P such that it is adjacent to both endpoints of P . If y sees an even number of edges of P then the graph contains an odd hole.

Proof by induction on: induction step

| y sees no other vertex odd hole (C_{n+1}) .

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Lemma

Let $P = \{v_1, \dots, v_n\}$ be a path on vertices where $n \geq 4$ is even, let y be a vertex not on P such that it is adjacent to both endpoints of P . If y and P sees an even number of edges of P then the graph contains an odd hole.

Proof by induction on: induction step

- | y sees no other vertex odd hole (C_{n+1}) .
- | y sees first and last edge IH on $P^0 = \{v_2, \dots, v_{n-1}\}$.

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Otherwise, if y does not see last edge, let $k < n - 1$ be largest index s.t. y sees v_j .

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- | If j odd, then $(v_j, \dots, v_n) \cup \{y\}$ odd hole.

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Let $P = (v_1, \dots, v_n)$ be a path on vertices where $n \geq 4$ is even, let y be a vertex not on P such that it is adjacent to both endpoints of P . If y sees an even number of edges of P then the graph contains an odd hole.

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Otherwise, if y does not see last edge, let $j < n - 1$ be largest index s.t. y sees v_j .

- | If j odd, then $(v_j, \dots, v_n) \cup (y, v_j)$ odd hole.
- | If j even, then $j \in 2 \mathbb{N}$! IH on $P^0 = (v_1, \dots, v_j)$.

Running Example Perfect Deletion (vc)

G[X [A] S perfect) G S perfect

Running Example Perfect Deletion (vc)

$G[X \setminus A] \setminus S_{\text{perfect}} \setminus G \setminus S_{\text{perfect}}$

- Compute disjoint bases A_i for $i \in [k+2j \times j+1]$, $A = \bigcup_i A_i$.

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- | Compute disjoint bases A_i for $i \in [k+2j \times j + 1]$, $A = \bigcup_i A_i$.
- | Consider kernel graph $G[X \setminus A]$.

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- | Compute disjoint bases A_i for $i \in [k+2] \times [j+1]$, $A = \bigcup_i A_i$.
- | Consider kernel graph $G[X \setminus A]$.
- | Consider solution S (red) s.t. $G[X \setminus A] \setminus S$ perfect.

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- | For contradiction, suppose $G \setminus S$ contains odd hole.

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- | For contradiction, suppose $G \setminus S$ contains odd hole.
- | Let C be an odd hole s.t. $|V(C) \cap (X \setminus A)|$ minimum.

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- | For contradiction, suppose $G \setminus S$ contains odd hole.
- | Let C be an odd hole s.t. $|V(C) \cap (X \setminus A)|$ minimum.
- | $|C| + |S| \geq 2|X \setminus A| + k$, there exists A_i outside $S \cap C$ (e.g. A_2).

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$G[X \setminus A]$ S perfect) G S perfect

- | As v not marked, $\text{inc}_{(G;X)}^4(v) = \prod_{y \in A_i} \text{inc}_{(G;X)}^4(y)$ over F_2 .

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- | Claim: some $u \in A_i$ sees even number of edges $\text{deg} = C - v$.

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- | Claim: some $u \in A_i$ sees even number of edges of C v .
- | By lemma, there exists odd hole C^0 that contradicts minimality of C .

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- | Claim: some $u \in A_i$ sees even number of edges of C v .
- | By lemma, there exists odd hole C^0 that contradicts minimality of C .
- | Hence $G \setminus S$ does not contain odd hole.

Running Example Perfect Deletion (vc)

Concrete example

| $X = \{x_1, \dots, x_6\}$ and v .

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- | y_2 sees 2 edges ~~α~~ .

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- | $Y = \{y_1, \dots, y_7\}$.
- | $\text{inc}_{(G;X)}^4(v) = \sum_{y \in Y} \text{inc}_{(G;X)}^4(y)$ over F_2 .
- | y_2 sees 2 edges of X .
- | $G[\{x_2, x_3, x_4, x_5, y_2\}]$ induces odd hole.

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Odd anti-holes can be dealt with in a similar fashion.

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Theorem

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We can generalize this incidence vector approach.

Meta-theorem

Definition (rank- c adjacencies)

Let $c \geq \mathbb{N}$. Graph property \mathcal{P} is characterized by rank- c adjacencies if the following holds:

For each graph H , for each vertex cover X of H , for each set $D \subseteq V(H) \setminus X$, for each $v \in V(H) \setminus (D \cup X)$, if

| $H \setminus D \in \mathcal{P}$, and

| $\text{inc}_{(H;X)}^c(v) = \prod_{u \in D} \text{inc}_{(H;X)}^c(u)$ when evaluated over \mathbb{F}_2 ,

then there exists $D^0 \subseteq D$ such that $H \setminus v \setminus (D \setminus D^0) \in \mathcal{P}$.

Meta-theorem

Theorem [Fomin et al. 2014]

If \mathcal{P} is a graph property such that:

- (i) \mathcal{P} is characterized by c adjacencies,
- (ii) every graph in \mathcal{P} contains at least one edge, and
- (iii) there is a polynomial $p: \mathbb{N} \rightarrow \mathbb{N}$ such that all graphs G that are vertex-minimal with respect to \mathcal{P} satisfy $|V(G)| \leq p(\text{vc}(G))$,

then \mathcal{P} -free Deletion parameterized by the vertex cover size x admits a polynomial kernel with $O((x + p(x))x^c)$ vertices.

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Theorem [Jansen & de Kroon]

If \mathcal{P} is a graph property such that:

- (i) \mathcal{P} is characterized by rank- c adjacencies,
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Meta-theorem

Fomin et al. [2014]

$:=$ all graphs that...	$c?$	-free deletion kernel
contain C_n for some n	1	$O(X)$ vrtcs
contain an odd cycle	2	$O(X ^3)$ vrtcs
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$:=$ all graphs that...	$c?$	-free deletion kernel
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Our results

$:=$ all graphs that...	rank- $c?$	-free deletion kernel
are not perfect	4	$O(jX^5)$ vrtcs
contain even holes	3	$O(jX^4)$ vrtcs
contain asteroidal triples	8	$O(jX^9)$ vrtcs
are not interval	8	$O(jX^9)$ vrtcs
contain a wheel	4	$O(jX^5)$ vrtcs

Meta-theorem

We gave a weaker sufficient condition for polynomial kernelization of \mathcal{H} -free Deletion (vc).

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χ := all graphs that...	rank- χ ?	χ -free deletion kernel
contain a wheel	4	$O(n^5)$ vrtcs

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$:=$ all graphs that...	rank- \mathcal{C} ?	\mathcal{C} -free deletion kernel
contain a wheel	4	$O(X ^5)$ vrtcs
contain a wheel besides W_4	no $\mathcal{C} \geq N$	no poly (NP $\not\subseteq$ coNP/poly)

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Open problems:

- | Is the meta-theorem tight now?

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Open problems:

- | Is the meta-theorem tight now?
- | Can the meta-theorem be used for Permutation Deletion or Comparability Deletion?